Dirac-Schrödinger equation for quark-antiquark bound states and derivation of its interaction kernel

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The four-dimensional Dirac-Schrödinger equation satisfied by quark-antiquark bound states is derived from Quantum Chromodynamics. Different from the Bethe-Salpeter equation, the equation derived is a kind of first-order differential equations of Schrödinger-type in the position space. Especially, the interaction kernel in the equation is given by two different closed expressions. One expression which contains only a few types of Green's functions is derived with the aid of the equations of motion satisfied by some kinds of Green's functions. Another expression which is represented in terms of the quark, antiquark and gluon propagators and some kinds of proper vertices is derived by means of the technique of irreducible decomposition of Green's functions. The kernel derived not only can easily be calculated by the perturbation method, but also provides a suitable basis for nonperturbative investigations. Furthermore, it is shown that the four-dimensinal Dirac-Schrödinger equation and its kernel can directly be reduced to rigorous three-dimensional forms in the equal-time Lorentz frame and the Dirac-Schrödinger equation can be reduced to an equivalent Pauli-Schrödinger equation which is represented in the Pauli spinor space. To show the applicability of the closed expressions derived and to demonstrate the equivalence between the two different expressions of the kernel, the t-channel and s-channel one gluon exchange kernels are chosen as an example to show how they are derived from the closed expressions. In addition, the connection of the Dirac-Schrödinger equation with the Bethe-Salpeter equation is discussed.

1. Introduction

It is the common recognition that the Bethe-Salpeter (B-S) equation which was proposed early in Refs.[1, 2] is a rigorous formalism for relativistic bound states. The prominent features of the equation are: (1) The equation is derived from the quantum field theory and hence set up on the firm dynamical basis; (2) The interaction kernel in the equation contains all the interactions taking place in the bound states and therefore the equation provides a possibility of exactly solving the problem of relativistic bound states; (3) The equation is elegantly formulated in a manifestly Lorentz-covariant form in the Minkowski space which allows us to discuss the equation in any coordinate frame. However, there are tremendous difficulties in practical applications of the equation, particularly, for solving the nuclear force in the nuclear physics and the quark confinement in hadron physics. One of the difficulties arises from the fact that the kernel in the equation was not given a closed form in the past. The kernel usually is defined as a sum of B-S (two-particle) irreducible Feynman diagrams each of which can only be individually determined by a perturbative calculation. This definition is, certainly, not suitable to investigate the subjects such as the nuclear force and the quark confinement which must necessarily be solved by a nonperturbative method. This is why, as said in Ref. [3]," The Bethe-Salpeter equation has not led to a real breakthrough in our understanding of the quark-quark force". Opposite to the conventional concept as commented in Ref. [4] that "The kernel K can not be given in closed form expression", we have derived a closed expression of the B-S kernel for quark-antiquark bound states in a recent publication [5]. The expression derived contains only a few types of Green's functions which not only are easily calculated by the perturbation method, but also suitable to be investigated by a certain nonperturbation approach. Another difficulty of solving the B-S equation was attributed to the four-dimensional nature of the equation because the relative time (or the relative energy) would lead to unphysical solutions. Therefore, many efforts were made in the past to recast the four-dimensional equation in three-dimensional ones in either approximate manners or exact versions such as the instantaneous approximation [6], the quasipotential approach [7-12] and the equal-time formalism [13-16].

As one knows, the four-dimensionally covariant B-S equation for a two-fermion system is ordinarily formulated in a second-order differential equation with respect to the space-time variables in the position space. This kind of equation has been shown to have unphysical solutions with the negative norm. It was pointed out in Ref. [17] that" The appearance of the negative-norm B-S amplitude is a quite common phenomenon in the B-S equation". A similar phenomenon was encountered in the Klein-Gordon (K-G) equation [18,19]

$$(\Box_x + m^2)\psi(x) = 0 \tag{1.1}$$

which was originally viewed as the wave equation satisfied by the single free fermion states. It is well-known that the K-G equation, as a second-order differential equation, has a solution with negative probability. This is because the wave function of the equation is determined not only by its initial value $\psi(0)$, but also by the initial value of the time-differential $\frac{\partial \psi}{\partial t}|_{t=0}$ which would possibly cause the solution to have negative probability [18, 19]. Nevertheless, the Dirac equation [18, 19]

$$(i\partial_x - m)\psi(x) = 0 \tag{1.2}$$

where $\partial_x = \gamma^\mu \partial_\mu^x$ has not the negative norm solution because it is a first-order differential equation. As widely recognized, the Dirac equation gives a correct description of the single free fermion states.

Analogous to the case of single fermion, the relativistic states for a two-fermion system may also be formulated by a set of first-order differential equations just as the Hamilton equation in Mechanics and the Maxwell equation in Electrodynamics which are equivalent to the second-order differential equations, i.e. the Lagrange equation and the D'Alembert equation respectively. Motivated by this idea, it was proposed in the literature [20-28] that the quark-antiquark bound system may be described by two coupled Dirac equations which are constructed in accord with the Dirac's Hamiltonian constraint formalism [29] such that [23-25]

$$[i\partial_{x_1} - m_1 - V_1(x_1, x_2)]\psi(x_1, x_2) = 0 \tag{1.3}$$

$$(i\partial_{x_2} - m_2 - V_2(x_1, x_2)]\psi(x_1, x_2) = 0 \tag{1.4}$$

where $\psi(x_1, x_2)$ denotes the two-fermion wave function, m_1 and m_2 are the masses of quark and antiquark, V_1 and V_2 stand for the effective potentials which are determined by the requirement of satisfying the Lorentz-invariance, the charge conjugation symmetry and a certain constraint (or say, compatibility) conditions. With a constraint imposed on the relative time, the above equations will be reduced to a three-dimensional eigenvalue equation.

As emphasized in the previous literature [23-25], Eqs. (1.3) and (1.4) are built up within the framework of relativistic quantum mechanics and the interaction potentials are given in a phenomenological way although they are inspired by the quantum field theory and, as demonstrated in Ref. [23], are linked with the corresponding B-S equation. Obviously, in order to understand the Dirac-type equations for the two-fermion system more precisely, it is necessary to give such equations an extensive investigation and an exact formulation from the viewpoint of quantum field theory. This just is the purpose of this paper. In this paper, we limit ourself to discuss the quark and antiquark ($q\overline{q}$) bound states. The results certainly suit to other two-fermion bound systems. First we derive two first-order differential equations for the quark-antiquark bound states from Quantum Chromodynamics (QCD) which describe the evolution of the bound state with the total (center of mass) time and the relative time respectively. These equations will be called Dirac-Schrödinger (D-S) equation because the Dirac equation is, in essence, the Schrödinger equation in the relativistic case which is identified with itself as the uniquely correct equation of describing the evolution of a quantum state with time in the quantum theory. Next, we concentrate our main attention on the interaction kernel appearing in the D-S equation. We are devoted to deriving a closed and explicit expression of the interaction kernel. The kernel will be derived by two different methods: one is to utilize equations of motion satisfied by the $q\overline{q}$ four-point Green's function and some other four-point Green's functions in which the gluon field is involved; another is to employ the technique of irreducible decomposition of the Green's functions involved in the D-S equation. The first method is similar to that proposed previously in Ref. [14]. The kernel derived by this method has a compact expression which contains only a few types of Green's functions. The kernel derived by the second method is expressed in terms of the quark, antiquark and gluon propagators and some kinds of three, four and five-line proper vertices and therefore exhibits a more specific structure of the kernel. Especially, the kernel derived can not only be easily calculated by the perturbation method, but also provides a suitable basis for nonperturbative investigations. The D-S equation and its interaction kernel mentioned above are Lorentz-covariant. We will show how this equation and its kernel are reduced to the exact three-dimensional forms given previously in Ref. [15] in the equal-time Lorentz frame. It is well-known that the D-S equation is represented in the Dirac spinor space. This equation actually is a coupled set of sixteen scalar equations. In practical applications, sometimes it is more convenient to reduce the D-S equation to the Pauli spinor space following the procedure proposed in Ref. [30]. By this procedure, we will obtain an equivalent Pauli-Schrödinger (P-S) equation represented in the Pauli spinor space from the D-S equation. In the P-S equation, the interaction Hamiltonian is explicitly given in a series expression which has an one-to-one correspondence with the perturbative expansion of the S-matrix. To illustrate the applicability

of the kernels derived and the equivalence between the aforementioned two different expressions of the kernel, we will show how the one-gluon exchange kernels in the D-S equation and the corresponding interaction Hamiltonian in the P-S equation can be derived from the closed expressions. Finally, we will discuss the relation between the D-S equation and the corresponding B-S equation.

The remainder of this paper is arranged as follows. In section 2, we will first derive two Dirac-type equations satisfied by the $q\overline{q}$ four-point Green's function. From these equations, the D-S equations obeyed by the B-S amplitudes will be derived by making use of the Lehmann representation of the Green's function [31]. Then, we will show how the four-dimensional D-S equation is reduced to the three-dimensional one. In section 3, the first explicit expression of the interaction kernel in the D-S equation will be derived by virtue of the equations of motion satisfied by the Green's functions involved in the Dirac-like equations. And, it will be shown how the closed expression of the exact three-dimensional kernel can be written out from the four-dimensional one. In section 4, the second expression of the interaction kernel will be derived by means of the technique of irreducible decomposition of the Green's functions. In section 5, the D-S equation will be reduced to the corresponding P-S equation. Section 6 will be used to give a brief derivation and description of the one-gluon exchange kernels. The last section serve to discuss the relation between the D-S equation and the corresponding B-S equation and to make some remarks. In Appendix A, we will describe the derivation of the equations of motion satisfied by the Green's functions which are necessary to be used in the derivation of the D-S equation and its interaction kernel. In Appendix B, the irreducible decomposition of the relevant Green's functions will be performed for the purpose of deriving the second expression of the kernel.

2. Derivation of the Dirac-Schrödinger equation

The Dirac-Schrödinger (D-S) equation satisfied by the $q\overline{q}$ bound states may be derived from the corresponding equation for the $q\overline{q}$ four-point Green's function which is defined in the Heisenberg picture as follows [32]

$$\mathcal{G}(x_1, x_2; y_1, y_2)_{\alpha\beta\rho\sigma} = \langle 0^+ \left| T\{N[\psi_\alpha(x_1)\psi_\beta^c(x_2)]N[\overline{\psi}_\rho(y_1)\overline{\psi}_\sigma^c(y_2)]\} \right| 0^- \rangle \tag{2.1}$$

where $\psi(x)$ and $\psi^c(x)$ are the quark and antiquark field operators respectively, $\overline{\psi}(x)$ and $\overline{\psi}^c(x)$ are their corresponding Dirac conjugates [18]

$$\psi^c(x) = C\overline{\psi}^T(x), \overline{\psi}^c(x) = -\psi^T(x)C^{-1} \tag{2.2}$$

here $C = i\gamma^2\gamma^0$ is the charge conjugation operator, $\mid 0^\pm \rangle$ denote the physical vacuum states, T symbolizes the time-ordering product and N designates the normal product which is defined by

$$N[\psi_\alpha(x_1)\psi_\beta^c(x_2)] = T[\psi_\alpha(x_1)\psi_\beta^c(x_2)] - \langle 0^+ \left| T[\psi_\alpha(x_1)\psi_\beta^c(x_2)] \right| 0^- \rangle \tag{2.3}$$

It is emphasized here that the above normal product can only be viewed as a definition in the Heisenberg picture. With the definition shown in Eq. (2.3), the Green's function in Eq. (2.1) may be represented as

$$\mathcal{G}(x_1, x_2; y_1, y_2)_{\alpha\beta\rho\sigma} = G(x_1, x_2; y_1, y_2)_{\alpha\beta\rho\sigma} + S_F^*(x_1 - x_2)_{\alpha\beta}\overline{S}_F^*(y_1 - y_2)_{\rho\sigma} \tag{2.4}$$

where

$$G(x_1, x_2; y_1, y_2)_{\alpha\beta\rho\sigma} = \langle 0^+ \left| T\{\psi_\alpha(x_1)\psi_\beta^c(x_2)\overline{\psi}_\rho(y_1)\overline{\psi}_\sigma^c(y_2)\} \right| 0^- \rangle \tag{2.5}$$

is the ordinary $q\overline{q}$ four-point Green's function [18],

$$S_F^*(x_1 - x_2)_{\alpha\beta} = \frac{1}{i}\langle 0^+ \left| T\{\psi_\alpha(x_1)\psi_\beta^c(x_2)\} \right| 0^- \rangle$$
$$= S_F(x_1 - x_2)_{\alpha\gamma}(C^{-1})_{\gamma\beta} = S_F^c(x_2 - x_1)_{\beta\lambda}C_{\lambda\alpha} \tag{2.6}$$

and

$$\overline{S}_F^*(y_1 - y_2)_{\rho\sigma} = \frac{1}{i}\langle 0^+ \left| T\{\overline{\psi}_\rho(y_1)\overline{\psi}_\sigma^c(y_2)\} \right| 0^- \rangle$$
$$= C_{\sigma\tau}S_F(y_2 - y_1)_{\tau\rho} = (C^{-1})_{\rho\delta}S_F^c(y_1 - y_2)_{\delta\sigma} \tag{2.7}$$

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in which

$$S_F(x_1 - x_2)_{\alpha\gamma} = \langle 0^+ | T\{\psi_\alpha(x_1)\overline{\psi}_\gamma(x_2)\} | 0^- \rangle \tag{2.8}$$

and

$$S_F^c(y_1 - y_2)_{\delta\sigma} = \langle 0^+ | T\{\psi_\delta^c(y_1)\overline{\psi}_\sigma^c(y_2)\} | 0^- \rangle \tag{2.9}$$

are the ordinary quark and antiquark propagators respectively [18]. It is clear that the propagators defined in Eqs. (2.6) and (2.7) are nonzero only for the quark and the antiquark which are of the same flavor. For the quark and antiquark of different flavors, the Green's function defined in Eq. (2.1) is reduced to the ordinary form shown in Eq. (2.5) since the second term on the right hand side (RHS) of Eq. (2.4) vanishes. In the case of the quark and antiquark of the same flavor, the normal product in Eq. (2.1) plays a role of excluding the contraction between the quark field and the antiquark one from the Green's function. Physically, this avoids the $q\overline{q}$ annihilation to break stability of a bound state. It would be pointed out that use of $\psi^c(x)$ other than $\overline{\psi}(x)$ to represent the antiquark field in this paper has an advantage that the antiquark field would behave as a quark one in the D-S equation so that the quark-antiquark equation formally is the same as the corresponding two-quark equation in the case that the quark and antiquark have different flavors.

The equations of motion which describe the variation of the $q\overline{q}$ four-point Green's function $G(x_1, x_2; y_1, y_2)$ with the coordinates x_1 and x_2 may easily be derived from the QCD generating functional as described in Appendix A. The results are

$$(i\partial_{x_1} - m_1)_{\alpha\gamma} G(x_1, x_2; y_1, y_2)_{\gamma\beta\rho\sigma} = \delta_{\alpha\rho}\delta^4(x_1 - y_1)S_F^c(x_2 - y_2)_{\beta\sigma}$$
$$+ C_{\alpha\beta}\delta^4(x_1 - x_2)\overline{S}_F^*(y_1 - y_2)_{\rho\sigma} - (\Gamma^{a\mu})_{\alpha\gamma}G_\mu^a(x_1 \mid x_1, x_2; y_1, y_2)_{\gamma\beta\rho\sigma} \tag{2.10}$$

$$(i\partial_{x_2} - m_2)_{\beta\lambda} G(x_1, x_2; y_1, y_2)_{\alpha\lambda\rho\sigma} = \delta_{\beta\sigma}\delta^4(x_2 - y_2)S_F(x_1 - y_1)_{\alpha\rho}$$
$$+ C_{\alpha\beta}\delta^4(x_1 - x_2)\overline{S}_F^*(y_1 - y_2)_{\rho\sigma} - (\overline{\Gamma}^{b\nu})_{\beta\lambda}G_\nu^b(x_2 \mid x_1, x_2; y_1, y_2)_{\alpha\lambda\rho\sigma} \tag{2.11}$$

in which

$$(\Gamma^{a\mu})_{\alpha\gamma} = g(\gamma^\mu T^a)_{\alpha\gamma}, \quad (\overline{\Gamma}^{b\nu})_{\beta\lambda} = g(\gamma^\nu \overline{T}^b)_{\beta\lambda} \tag{2.12}$$

where g is the coupling constant, $T^a = \frac{\lambda^a}{2}$ and $\overline{T}^a = -\lambda^{a*}/2$ are the quark and antiquark color matrices respectively,

$$G_\mu^a(x_i \mid x_1, x_2; y_1, y_2)_{\alpha\beta\rho\sigma}$$
$$= \langle 0^+ | T\{\mathbf{A}_\mu^a(x_i)\psi_\alpha(x_1)\psi_\beta^c(x_2)\overline{\psi}_\rho(y_1)\overline{\psi}_\sigma^c(y_2)\} | 0^- \rangle \tag{2.13}$$

with $i = 1, 2$ are the new four-point Green's function including a gluon field in it and the propagators were defined in Eqs. (2.6)-(2.9). It would be noted here that the terms related to $\overline{S}_F^*(y_1 - y_2)$ in Eqs. (2.10) and (2.11) are absent when the quark and the antiquark have different flavors. The equations of motion satisfied by the Green's function defined in Eq. (2.1) may be found by substituting the relation in Eq. (2.4) into Eqs. (2.10) and (2.11) and by making use of the following equations as mentioned in Appendix A

$$[(i\partial_{x_1} - m_1)_{\alpha\gamma} S_F^*(x_1 - x_2)_{\gamma\beta} = -C_{\alpha\beta}\delta^4(x_1 - x_2) - (\Gamma^{a\mu})_{\alpha\gamma}\Lambda_\mu^{a*}(x_1 \mid x_1, x_2)_{\gamma\beta}$$
$$[(i\partial_{x_2} - m_2)_{\beta\lambda} S_F^*(x_1 - x_2)_{\alpha\lambda} = -C_{\alpha\beta}\delta^4(x_1 - x_2) - (\overline{\Gamma}^{b\nu})_{\beta\lambda}\Lambda_\nu^{b*}(x_2 \mid x_1, x_2)_{\alpha\lambda} \tag{2.14}$$

where

$$\Lambda_\mu^{a*}(x_i \mid x_1, x_2)_{\alpha\beta} = \frac{1}{i}\langle 0^+ | T\{\mathbf{A}_\mu^a(x_i)\psi_\alpha(x_1)\psi_\beta^c(x_2)\} | 0^- \rangle \tag{2.15}$$

with $i = 1, 2$ are a kind of quark-antiquark-gluon Green's function. The results are

$$(i\partial_{x_1} - m_1)_{\alpha\gamma} \mathcal{G}_{\gamma\beta\rho\sigma}(x_1, x_2; y_1, y_2) = \delta_{\alpha\rho}\delta^4(x_1 - y_1)S_F^c(x_2 - y_2)_{\beta\sigma}$$
$$- (\Gamma^{a\mu})_{\alpha\gamma}\mathcal{G}_\mu^a(x_1 \mid x_1, x_2; y_1, y_2)_{\gamma\beta\rho\sigma} \tag{2.16}$$

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and

$$(i\partial_{x_2} - m_2)_{\beta\lambda}\mathcal{G}_{\alpha\lambda\rho\sigma}(x_1, x_2; y_1, y_2) = \delta_{\beta\sigma}\delta^4(x_2 - y_2)S_F(x_1 - y_1)_{\alpha\rho}$$
$$-(\overline{\Gamma}^{b\nu})_{\beta\lambda}\mathcal{G}^b_\nu(x_2 \mid x_1, x_2; y_1, y_2)_{\alpha\lambda\rho\sigma} \tag{2.17}$$

where

$$\mathcal{G}^a_\mu(x_i \mid x_1, x_2; y_1, y_2)_{\alpha\beta\rho\sigma} = \langle 0^+ \left| T[\{N[\mathbf{A}^a_\mu(x_i)\psi_\alpha(x_1)\psi^c_\beta(x_2)]N[\overline{\psi}_\rho(y_1)\overline{\psi}^c_\sigma(y_2)]\} \right| 0^- \rangle$$
$$= G^a_\mu(x_i \mid x_1, x_2; y_1, y_2)_{\alpha\beta\rho\sigma} + \Lambda^{a*}_\mu(x_i \mid x_1, x_2)_{\alpha\beta}\overline{S}^*_F(y_1 - y_2)_{\rho\sigma} \tag{2.18}$$

here $i = 1, 2$. In the above, the normal products defined in the same way as that in Eq. (2.3). In comparison of Eqs. (2.16) and (2.17) with Eqs. (2.10) and (2.11), we see, the terms related to $\overline{S}^*_F(y_1 - y_2)$ in Eqs. (2.10) and (2.11) disappear in Eqs. (2.16) and (2.17). Therefore, the equations (2.16) and (2.17) formally are the same as given in the case that the quark and the antiquark are of different flavors.

Multiplying Eqs. (2.16) and (2.17) with the matrices γ^0_1 and γ^0_2 respectively, we have

$$[i\tfrac{\partial}{\partial t_1} - h^{(1)}(\overrightarrow{x_1})]_{\alpha\gamma}\mathcal{G}_{\gamma\beta\rho\sigma}(x_1, x_2; y_1, y_2)$$
$$= (\gamma^0_1)_{\alpha\rho}\delta^4(x_1 - y_1)S^c_F(x_2 - y_2)_{\beta\sigma} + \mathcal{G}^{(1)}_{\alpha\beta\rho\sigma}(x_1, x_2; y_1, y_2) \tag{2.19}$$

and

$$[i\tfrac{\partial}{\partial t_2} - h^{(2)}(\overrightarrow{x_2})]_{\beta\lambda}\mathcal{G}_{\alpha\lambda\rho\sigma}(x_1, x_2; y_1, y_2)$$
$$= (\gamma^0_2)_{\beta\sigma}\delta^4(x_2 - y_2)S_F(x_1 - y_1)_{\alpha\rho} + \mathcal{G}^{(2)}_{\alpha\beta\rho\sigma}(x_1, x_2; y_1, y_2) \tag{2.20}$$

where

$$h^{(i)}(\overrightarrow{x_i}) = -i\overrightarrow{\alpha_i} \cdot \nabla_{\overrightarrow{x_i}} + m_i\gamma^0_i \tag{2.21}$$

is the i-th free fermion Hamiltonian,

$$\mathcal{G}^{(1)}_{\alpha\beta\rho\sigma}(x_1 \mid x_1, x_2; y_1, y_2) = (\Omega^{a\mu}_1)_{\alpha\gamma}\mathcal{G}^a_\mu(x_1 \mid x_1, x_2; y_1, y_2)_{\gamma\beta\rho\sigma}$$
$$\mathcal{G}^{(2)}_{\alpha\beta\rho\sigma}(x_2 \mid x_1, x_2; y_1, y_2) = (\Omega^{b\nu}_2)_{\beta\lambda}\mathcal{G}^b_\nu(x_2 \mid x_1, x_2; y_1, y_2)_{\alpha\lambda\rho\sigma} \tag{2.22}$$

here

$$\Omega^{a\mu}_1 = -g\gamma^0_1\gamma^\mu_1 T^a_1, \ \Omega^{b\nu}_2 = -g\gamma^0_2\gamma^\nu_2 \overline{T}^b_2 \tag{2.23}$$

As will be proved in section 4, the Green's functions $\mathcal{G}^a_\mu(x_i \mid x_1, x_2; y_1, y_2)$ are B-S reducible, therefore, we can write

$$\mathcal{G}^{(i)}(x_i \mid x_1, x_2; y_1, y_2) = \int d^4z_1 d^4z_2 K^{(i)}(x_1, x_2; z_1, z_2)\mathcal{G}(z_1, z_2; y_1, y_2) \tag{2.24}$$

where $K^{(i)}(x_1, x_2; z_1, z_2)$ $(i = 1, 2)$ are just the interaction kernels. With the expression given in the above, Eqs. (2.19) and (2.20) can be represented as

$$[i\tfrac{\partial}{\partial t_1} - h^{(1)}(\overrightarrow{x_1})]_{\alpha\gamma}\mathcal{G}_{\gamma\beta\rho\sigma}(x_1, x_2; y_1, y_2) = (\gamma^0_1)_{\alpha\rho}\delta^4(x_1 - y_1)S^c_F(x_2 - y_2)_{\beta\sigma}$$
$$+ \int d^4z_1 d^4z_2 K^{(1)}(x_1, x_2; z_1, z_2)_{\alpha\beta\lambda\tau}\mathcal{G}_{\lambda\tau\rho\sigma}(z_1, z_2; y_1, y_2) \tag{2.25}$$

and

$$[i\tfrac{\partial}{\partial t_2} - h^{(2)}(\overrightarrow{x_2})]_{\beta\lambda}\mathcal{G}_{\alpha\lambda\rho\sigma}(x_1, x_2; y_1, y_2) = (\gamma^0_2)_{\beta\sigma}\delta^4(x_2 - y_2)S_F(x_1 - y_1)_{\alpha\rho}$$
$$+ \int d^4z_1 d^4z_2 K^{(2)}(x_1, x_2; z_1, z_2)_{\alpha\beta\lambda\tau}\mathcal{G}_{\lambda\tau\rho\sigma}(z_1, z_2; y_1, y_2) \tag{2.26}$$

From the above two equations, we may obtain two equivalent equations: one describes the evolution of the Green's function $\mathcal{G}(z_1, z_2; y_1, y_2)$ with the center of mass time; another describes the evolution of the Green's function with

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the relative time. The first equation is given by summing up the two equations in Eqs. (2.25) and (2.26). Introducing the cluster coordinates

$$
\begin{aligned}
&X = \eta_1 x_1 + \eta_2 x_2,\ x = x_1 - x_2; \\
&Y = \eta_1 y_1 + \eta_2 y_2,\ y = y_1 - y_2, \\
&\eta_i = \frac{m_i}{m_1 + m_2}, i = 1, 2
\end{aligned}
\tag{2.27}
$$

the equation may be represented in the matrix notation as

$$
\begin{aligned}
&[i\frac{\partial}{\partial t} - H_0(\vec{X}, \vec{x})]\mathcal{G}(X - Y, x, y) \\
&= S(X - Y, x, y) + \int d^4 Z d^4 z K(X - Z, x, z)\mathcal{G}(Z - Y, z, y)
\end{aligned}
\tag{2.28}
$$

where

$$
S(X - Y, x, y) = \delta^4(x_1 - y_1)\gamma_1^0 S_F^c(x_2 - y_2) + \delta^4(x_2 - y_2)\gamma_2^0 S_F(x_1 - y_1)
\tag{2.29}
$$

$$
H_0(\vec{X}, \vec{x}) = h^{(1)}(\vec{x}_1) + h^{(2)}(\vec{x}_2)
\tag{2.30}
$$

is the total free Hamiltonian,

$$
K(X - Z, x, z) = \sum_{i=1}^{2} K^{(i)}(X - Z, x, z)
\tag{2.31}
$$

is the total interaction kernel and $t = X_0$ is the center of mass time. In the above, the translation-invariance of the Green's function and the interaction kernel has been considered.

The second equation mentioned above is given by subtracting the equation in Eq. (2.26) with weight η_1 from the equation in Eq. (2.25) with weight η_2

$$
\begin{aligned}
&[i\frac{\partial}{\partial \tau} - \overline{H}_0(\vec{X}, \vec{x})]\mathcal{G}(X - Y, x, y) \\
&= \overline{S}(X - Y, x, y) + \int d^4 Z d^4 z \overline{K}(X - Z, x, z)\mathcal{G}(Z - Y, z, y)
\end{aligned}
\tag{2.32}
$$

where

$$
\overline{S}(X - Y, x, y) = \eta_2 \delta^4(x_1 - y_1)\gamma_1^0 S_F^c(x_2 - y_2) - \eta_1 \delta^4(x_2 - y_2)\gamma_2^0 S_F(x_1 - y_1)
\tag{2.33}
$$

$$
\overline{H}_0(\vec{X}, \vec{x}) = \eta_2 h^{(1)}(\vec{x}_1) - \eta_1 h^{(2)}(\vec{x}_2)
\tag{2.34}
$$

which is the relative Hamiltonian,

$$
\overline{K}(X - Z, x, z) = \eta_2 K^{(1)}(X - Z, x, z) - \eta_1 K^{(2)}(X - Z, x, z)
\tag{2.35}
$$

which is the relative kernel and $\tau = x_0$ is the relative time.

By virtue of the well-known Lehmann representation of the Green's function $\mathcal{G}(z_1, z_2; y_1, y_2)$ [31], one may derived the equations satisfied by the B-S amplitude from the above equations. The Lehmann representation as shown below can easily be written out by the procedure of inserting the complete set of the $q\bar{q}$ bound states into the Green's function $\mathcal{G}(z_1, z_2; y_1, y_2)$ denoted in Eq. (2.1), then considering the translation-invariance property of the Green's function and finally employing the integral representation of the step function,

$$
\begin{aligned}
&\mathcal{G}(X - Y, x, y) = \sum_n \frac{i}{(2\pi)^4} \int d^4 Q_n e^{-iQ_n(X-Y)} \\
&\times \frac{1}{2\omega_n}\{\frac{\chi_{Q_n}(x)\overline{\chi}_{Q_n}(y)}{Q_n^0 - \omega_n + i\epsilon} - \frac{\chi_{-Q_n}^+(y)\overline{\chi}_{-Q_n}^+(x)}{Q_n^0 + \omega_n - i\epsilon}\}
\end{aligned}
\tag{2.36}
$$

where

$$\chi_{Q_n}(X, x) = e^{-iQ_n X} \chi_{Q_n}(x) = \langle 0^+ | N\{\psi(x_1)\psi^c(x_2)\} | n \rangle$$
$$\overline{\chi}_{Q_n}(Y, y) = e^{iQ_n Y} \overline{\chi}_{Q_n}(y) = \langle n | N\{\overline{\psi}(y_1)\overline{\psi}^c(y_2)\} | 0^- \rangle \tag{2.37}$$

are the B-S amplitudes describing the bound state and ω_n is the energy of the state $| n \rangle$. Upon substituting Eq. (2.36) into Eqs. (2.28) and (2.32), then taking the limit: $\lim_{Q_n^0 \to \omega_n}(Q_n^0 - \omega_n)$ and finally performing the integration: $\int d^4 Y e^{-iPY}$, one can find that

$$[i\frac{\partial}{\partial t} - H_0(\overrightarrow{X}, \overrightarrow{x})]\chi_{P\varsigma}(X, x) = \int d^4 Y d^4 y K(X - Y, x, y)\chi_{P\varsigma}(Y, y) \tag{2.38}$$

and

$$[i\frac{\partial}{\partial \tau} - \overline{H}_0(\overrightarrow{X}, \overrightarrow{x})]\chi_{P\varsigma}(X, x) = \int d^4 Y d^4 y \overline{K}(X - Y, x, y)\chi_{P\varsigma}(Y, y) \tag{2.39}$$

where the subscript ς in the B-S amplitude designates the other quantum numbers of a bound state. In the above derivation, the fact that the functions $S(X - Y, x, y)$ and $\overline{S}(X - Y, x, y)$ have no the bound state poles has been considered. Eqs. (2.38) and (2.39) are just the wanted D-S equations satisfied by the B-S amplitudes. Eq. (2.38) describes the evolution of the $q\overline{q}$ bound state with the center of mass time t, while, Eq. (2.39) describes the evolution of the $q\overline{q}$ bound state with the relative time τ. Clearly, both of the equations are all the first-order differential equations whose solutions are only determined by the initial amplitudes at the origin of times.

Considering the translation-invariance of the B-S amplitude and the kernels as shown in Eq. (2.37) and in the following

$$K(X - Y, x, y) = \int \frac{d^4 Q}{(2\pi)^4} e^{-iQ(X-Y)} K(Q, x, y)$$
$$\overline{K}(X - Y, x, y) = \int \frac{d^4 Q}{(2\pi)^4} e^{-iQ(X-Y)} \overline{K}(Q, x, y) \tag{2.40}$$

one can obtain from Eqs. (2.38) and (2.39) the equations satisfied by the amplitude which describes the internal motion of the $q\overline{q}$ bound system

$$[E - H_0(\overrightarrow{P}, \overrightarrow{x})]\chi_{P\varsigma}(x) = \int d^4 y K(P, x, y)\chi_{P\varsigma}(y) \tag{2.41}$$

$$[i\frac{\partial}{\partial \tau} - \overline{H}_0(\overrightarrow{P}, \overrightarrow{x})]\chi_{P\varsigma}(x) = \int d^4 y \overline{K}(P, x, y)\chi_{P\varsigma}(y) \tag{2.42}$$

Furthermore, in view of the Fourier transformation

$$\chi_{P\varsigma}(x) = \int \frac{d^4 q}{(2\pi)^4} e^{-iqx} \chi_{P\varsigma}(q)$$
$$K(P, x, y) = \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} e^{-iqx+iky} K(P, q, k), \tag{2.43}$$

Eqs. (2.41) and (2.42) will immediately be transformed into the momentum space

$$[E - H_0(\overrightarrow{P}, \overrightarrow{q})]\chi_{P\varsigma}(q) = \int \frac{d^4 k}{(2\pi)^4} K(P, q, k)\chi_{P\varsigma}(k) \tag{2.44}$$

$$[q_0 - \overline{H}_0(\overrightarrow{P}, \overrightarrow{q})]\chi_{P\varsigma}(q) = \int \frac{d^4 k}{(2\pi)^4} \overline{K}(P, q, k)\chi_{P\varsigma}(k) \tag{2.45}$$

where

$$H_0(\overrightarrow{P}, \overrightarrow{q}) = h^{(1)}(\overrightarrow{p_1}) + h^{(2)}(\overrightarrow{p_2})$$
$$\overline{H}_0(\overrightarrow{P}, \overrightarrow{q}) = \eta_2 h^{(1)}(\overrightarrow{p}_1) - \eta_1 h^{(2)}(\overrightarrow{p}_2) \tag{2.46}$$

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in which

$$h^{(i)}(\overrightarrow{p_i}) = \overrightarrow{p_i} \cdot \overrightarrow{\alpha_i} + m_i\gamma_i^0, \ i = 1, 2,$$
$$P = p_1 + p_2, \ q = \eta_2 p_1 - \eta_1 P_2, \ k = \eta_2 q_1 - \eta_1 q_2 \tag{2.47}$$

here q_i and p_i are the initial state and the final state momenta for i-th single particle. Both of the above equations are identified themselves with the eigenvalue equations for the $q\overline{q}$ bound system.

The D-S equations given in Eqs. (2.41) and (2.42) or (2.44) and (2.45) are Lorentz-covariant. They allow us to investigate the $q\overline{q}$ bound states in any coordinate frame. Based on the space-like property of a bound state, it is admissible to discuss the bound state in a special equal-time Lorentz frame. In this frame, the $q\overline{q}$ four-point Green's function becomes

$$\mathcal{G}(\overrightarrow{x_1},\overrightarrow{x_2}; \overrightarrow{y_1}, \overrightarrow{y_2}; t_1 - t_2) = \langle 0^+ \left| T\{N[\psi(\overrightarrow{x_1}, t_1)\psi^c(\overrightarrow{x_2}, t_1)]N[\overline{\psi}(\overrightarrow{y_1}, t_2)\overline{\psi}^c(\overrightarrow{y_2}, t_2)]\} \right| 0^- \rangle \tag{2.48}$$

The equation in Eq. (2.28) is now reduced to

$$[i\frac{\partial}{\partial t_1} - H_0(\overrightarrow{x_1},\overrightarrow{x_2})]\mathcal{G}(\overrightarrow{x_1},\overrightarrow{x_2}; \overrightarrow{y_1}, \overrightarrow{y_2}; t_1 - t_2)$$
$$= \delta(t_1 - t_2)S(\overrightarrow{x_1},\overrightarrow{x_2}; \overrightarrow{y_1}, \overrightarrow{y_2}) + \int d^3z_1 d^3z_2 dt_z K(\overrightarrow{x_1},\overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}; t_1 - t_z)\mathcal{G}(\overrightarrow{z_1},\overrightarrow{z_2}; \overrightarrow{y_1}, \overrightarrow{y_2}; t_z - t_2) \tag{2.49}$$

where

$$S(\overrightarrow{x_1},\overrightarrow{x_2}; \overrightarrow{y_1}, \overrightarrow{y_2}) = \delta^3(\overrightarrow{x_1} - \overrightarrow{y_1})\gamma_1^0 S_F^c(\overrightarrow{x_2} - \overrightarrow{y_2}) + \delta^3(\overrightarrow{x_2} - \overrightarrow{y_2})\gamma_2^0 S_F(\overrightarrow{x_1} - \overrightarrow{y_1}) \tag{2.50}$$

in which

$$S_F(\overrightarrow{x_1} - \overrightarrow{y_1}) = \frac{1}{i}\langle 0^+ \left| T[\psi(\overrightarrow{x_1}, t_1)[\overline{\psi}(\overrightarrow{y_1}, t_1)] \right| 0^- \rangle$$
$$S_F^c(\overrightarrow{x_2} - \overrightarrow{y_2}) = \langle 0^+ \left| T[\psi^c(\overrightarrow{x_2}, t_2)\overline{\psi}^c(\overrightarrow{y_2}, t_2)] \right| 0^- \rangle \tag{2.51}$$

are the equal-time quark and antiquark propagators respectively which are actually independent of time due to the translation-invariance property. By the Fourier transformations

$$\mathcal{G}(\overrightarrow{x_1},\overrightarrow{x_2}; \overrightarrow{y_1}, \overrightarrow{y_2}; t_1 - t_2) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} e^{iE(t_1 - t_2)}\mathcal{G}(\overrightarrow{x_1},\overrightarrow{x_2}; \overrightarrow{y_1}, \overrightarrow{y_2}; E)$$
$$K(\overrightarrow{x_1},\overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}; t_1 - t_2) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} e^{iE(t_1 - t_2)}K(\overrightarrow{x_1},\overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}; E) \tag{2.52}$$

Eq. (2.49) will be represented as [15]

$$[E - H_0(\overrightarrow{x_1},\overrightarrow{x_2})]\mathcal{G}(\overrightarrow{x_1},\overrightarrow{x_2}; \overrightarrow{y_1}, \overrightarrow{y_2}; E)$$
$$= S(\overrightarrow{x_1},\overrightarrow{x_2}; \overrightarrow{y_1}, \overrightarrow{y_2}) + \int d^3z_1 d^3z_2 K(\overrightarrow{x_1},\overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}; E)\mathcal{G}(\overrightarrow{z_1},\overrightarrow{z_2}; \overrightarrow{y_1}, \overrightarrow{y_2}; E) \tag{2.53}$$

This just is the three-dimensional equation satisfied by the $q\overline{q}$ four-point Green's function defined in the equal-time Lorentz frame.

In the equal-time frame, the relative time of the $q\overline{q}$ system is zero. Therefore, the equation in Eq. (2.32) becomes meaningless. We are left with only the equation given in Eq. (2.53). The Lehmann representation of the Green's function $\mathcal{G}(\overrightarrow{x_1},\overrightarrow{x_2}; \overrightarrow{y_1}, \overrightarrow{y_2}; t)$ is still represented in Eq. (2.36) except that the four-dimensional relative coordinates x and y in the B-S amplitudes are now replaced by the three-dimensional ones $\overrightarrow{x}$ and $\overrightarrow{y}$. Substituting such a Lehmann representation into Eq. (2.53), by the same procedure as stated in Eqs. (2.36)-(2.38), one may obtain a D-S equation represented in the three-dimensional coordinate space such that

$$[E - H_0(\overrightarrow{X}, \overrightarrow{x})]\chi_{P\varsigma}(\overrightarrow{X}, \overrightarrow{x}) = \int d^3Y d^3y K(\overrightarrow{X} - \overrightarrow{Y}, \overrightarrow{x}, \overrightarrow{y})\chi_{P\varsigma}(\overrightarrow{Y}, \overrightarrow{y}) \tag{2.54}$$

where

$$\chi_{P\varsigma}(\overrightarrow{X}, \overrightarrow{x}) = e^{i\overrightarrow{P}\cdot\overrightarrow{X}}\chi_{P\varsigma}(\overrightarrow{x}) \tag{2.55}$$

Apparently, in the momentum space, Eq. (2.54) becomes [15]

$$[E - H_0(\overrightarrow{P}, \overrightarrow{q})]\chi_{P\varsigma}(\overrightarrow{q}) = \int \frac{d^3k}{(2\pi)^3} K(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k}; E)\chi_{P\varsigma}(\overrightarrow{k}) \tag{2.56}$$

This is precisely the three-dimensional D-S equation satisfied by the amplitude $\chi_{P\varsigma}(\overrightarrow{k})$ which describes the internal motion of the $q\overline{q}$ bound system and may be used to solve the eigenvalue problem for the system. This equation is rigorous because the retardation effect is completely included in the kernel of the equation.

3. Derivation of the interaction kernel

In this section, the interaction kernel in the D-S equation will be derived by making use of equations of motion which describe the variation of the Green's functions with coordinates y_1 and y_2. The equations satisfied by the Green's function $G(x_1, x_2; y_1, y_2)$ are derived in Appendix A and can directly be written out from Eqs. (A.22) and (A.31) by setting the source J to be zero. The result is

$$
\begin{aligned}
&G(x_1, x_2; y_1, y_2)_{\alpha\beta\tau\sigma}(i\overleftarrow{\partial}_{y_1} + m_1)_{\tau\rho} = -\delta_{\alpha\rho}\delta^4(x_1 - y_1)S_F^c(x_2 - y_2)_{\beta\sigma} \\
&-C_{\rho\sigma}\delta^4(y_1 - y_2)S_F^*(x_1 - x_2)_{\alpha\beta} + G_\mu^a(y_1 \mid x_1, x_2; y_1, y_2)_{\alpha\beta\tau\sigma}(\Gamma^{a\mu})_{\tau\rho}
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
&G(x_1, x_2; y_1, y_2)_{\alpha\beta\rho\delta}(i\overleftarrow{\partial}_{y_2} + m_2)_{\delta\sigma} = -\delta_{\beta\sigma}\delta^4(x_2 - y_2)S_F(x_1 - y_1)_{\alpha\rho} \\
&-C_{\rho\sigma}\delta^4(y_1 - y_2)S_F^*(x_1 - x_2)_{\alpha\beta} + G_\nu^b(y_2 \mid x_1, x_2; y_1, y_2)_{\alpha\beta\rho\delta}(\overline{\Gamma}^{b\nu})_{\delta\sigma}
\end{aligned}
\tag{3.2}
$$

where $G_\mu^a(y_i \mid x_1, x_2; y_1, y_2)$ ($i = 1, 2$) were defined in Eq. (2.13) with the replacement of x_i by y_i.

Substituting the relation in Eq. (2.4) into the above two equations and employing the following equations obeyed by the propagator $\overline{S}_F^*(y_1 - y_2)$ whose derivation was mentioned in Appendix A

$$
\begin{aligned}
&\overline{S}_F^*(y_1 - y_2)_{\tau\sigma}(i\overleftarrow{\partial}_{y_1} + m_1)_{\tau\rho} = C_{\rho\sigma}\delta^4(y_1 - y_2) + \overline{\Lambda}_\nu^{b*}(y_1 \mid y_1, y_2)_{\tau\sigma}(\Gamma^{b\nu})_{\tau\rho} \\
&\overline{S}_F^*(y_1 - y_2)_{\rho\delta}(i\overleftarrow{\partial}_{y_2} + m_2)_{\delta\sigma} = C_{\rho\sigma}\delta^4(y_1 - y_2) + \overline{\Lambda}_\nu^{b*}(y_2 \mid y_1, y_2)_{\rho\delta}(\overline{\Gamma}^{b\nu})_{\delta\sigma}
\end{aligned}
\tag{3.3}
$$

where

$$
\overline{\Lambda}_\nu^{b*}(x_i \mid y_1, y_2)_{\tau\sigma} = \frac{1}{i}\left\langle 0^+ \left| T\{\mathbf{A}_\nu^b(x_i)\overline{\psi}_\tau(y_1)\overline{\psi}_\sigma^c(y_2)\} \right| 0^- \right\rangle,
\tag{3.4}
$$

it is easy to find

$$
\begin{aligned}
&\mathcal{G}(x_1, x_2; y_1, y_2)_{\alpha\beta\tau\sigma}(i\overleftarrow{\partial}_{y_1} + m_1)_{\tau\rho} = -\delta_{\alpha\rho}\delta^4(x_1 - y_1)S_F^c(x_2 - y_2)_{\beta\sigma} \\
&+\mathcal{G}_\mu^a(y_1 \mid x_1, x_2; y_1, y_2)_{\alpha\beta\tau\sigma}(\Gamma^{a\mu})_{\tau\rho}
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
&\mathcal{G}(x_1, x_2; y_1, y_2)_{\alpha\beta\rho\delta}(i\overleftarrow{\partial}_{y_2} + m_2)_{\delta\sigma} = -\delta_{\beta\sigma}\delta^4(x_2 - y_2)S_F(x_1 - y_1)_{\alpha\rho} \\
&+\mathcal{G}_\nu^b(y_2 \mid x_1, x_2; y_1, y_2)_{\alpha\beta\rho\delta}(\overline{\Gamma}^{b\nu})_{\delta\sigma}
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
\mathcal{G}_\mu^a(y_i \mid x_1, x_2; y_1, y_2)_{\alpha\beta\rho\sigma} &= \left\langle 0^+ \left| T\{N[\psi_\alpha(x_1)\psi_\beta^c(x_2)]N[\mathbf{A}_\mu^a(y_i)\overline{\psi}_\tau(y_1)\overline{\psi}_\sigma^c(y_2)]\} \right| 0^- \right\rangle \\
&= G_\mu^a(y_i \mid x_1, x_2; y_1, y_2)_{\alpha\beta\rho\sigma} + S_F^*(x_1 - x_2)_{\alpha\beta}\overline{\Lambda}_\mu^{a*}(y_i \mid y_1, y_2)_{\rho\sigma}
\end{aligned}
\tag{3.7}
$$

here $i = 1, 2$.

To derive the interaction kernel, we also need equations obeyed by the Green's functions $\mathcal{G}_\mu^a(x_i \mid x_1, x_2; y_1, y_2)$. According to the description given in Appendix A, the equations for the Green's functions $G_\mu^a(x_i \mid x_1, x_2; y_1, y_2)$ can be derived by differentiating Eqs. (A.22) and (A.31) with respect to the source $J^{a\mu}(x_i)$ and then setting $J = 0$. The result is

$$
\begin{aligned}
&G_\mu^a(x_i \mid x_1, x_2; y_1, y_2)_{\alpha\beta\tau\sigma}(i\overleftarrow{\partial}_{y_1} + m_1)_{\tau\rho} = -\delta_{\alpha\rho}\delta^4(x_1 - y_1)\Lambda_\mu^{\mathbf{c}a}(x_i \mid x_2, y_2)_{\beta\sigma} \\
&-C_{\rho\sigma}\delta^4(y_1 - y_2)\Lambda_\mu^{a*}(x_i \mid x_1, x_2)_{\alpha\beta} + G_{\mu\nu}^{ab}(x_i, y_1 \mid x_1, x_2; y_1, y_2)_{\alpha\beta\tau\sigma}(\Gamma^{a\mu})_{\tau\rho}
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
&G_\mu^a(x_i \mid x_1, x_2; y_1, y_2)_{\alpha\beta\rho\delta}(i\overleftarrow{\partial}_{y_2} + m_2)_{\delta\sigma} = -\delta_{\beta\sigma}\delta^4(x_2 - y_2)\Lambda_\mu^a(x_i \mid x_1, y_1)_{\alpha\rho} \\
&-C_{\rho\sigma}\delta^4(y_1 - y_2)\Lambda_\mu^{a*}(x_i \mid x_1, x_2)_{\alpha\beta} + G_{\mu\nu}^{ab}(x_i, y_2 \mid x_1, x_2; y_1, y_2)_{\alpha\beta\rho\delta}(\overline{\Gamma}^{b\nu})_{\delta\sigma}
\end{aligned}
\tag{3.9}
$$

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where

$$
\begin{aligned}
\Lambda_\mu^a(x_i \mid x_1, y_1)_{\gamma\rho} &= \tfrac{1}{i}\langle 0^+ \left| T[\mathbf{A}_\mu^a(x_i)\psi_\gamma(x_1)\overline{\psi}_\rho(y_1)] \right| 0^- \rangle, \\
\Lambda_\mu^{\mathbf{c}a}(x_i \mid x_2, y_2)_{\lambda\sigma} &= \tfrac{1}{i}\langle 0^+ \left| T[\mathbf{A}_\mu^a(x_i)\psi_\lambda^c(x_2)\overline{\psi}_\sigma^c(y_2)] \right| 0^- \rangle, \\
\Lambda_\mu^{a*}(x_i \mid y_1, y_2)_{\tau\sigma} &= \tfrac{1}{i}\langle 0^+ \left| T[\mathbf{A}_\mu^a(x_i)\psi_\tau(y_1)\psi_\sigma^c(y_2)] \right| 0^- \rangle
\end{aligned}
\tag{3.10}
$$

and

$$
G_{\mu\nu}^{ab}(x_i, y_j \mid x_1, x_2; y_1, y_2)_{\alpha\beta\rho\sigma} = \langle 0^+ \left| T\{\mathbf{A}_\mu^a(x_i)\mathbf{A}_\nu^b(y_j)\psi_\alpha(x_1)\psi_\beta^c(x_2)\overline{\psi}_\rho(y_1)\overline{\psi}_\sigma^c(y_2)\} \right| 0^- \rangle \tag{3.11}
$$

here $i, j = 1, 2$. On inserting the relation in Eq. (2.18) into Eqs. (3.8) and (3.9) and utilizing the equations in Eq. (3.3), we are led to

$$
\begin{aligned}
&\mathcal{G}_\mu^a(x_i \mid x_1, x_2; y_1, y_2)_{\alpha\beta\tau\sigma}(i\overset{\leftarrow}{\partial}_{y_1} + m_1)_{\tau\rho} = -\delta_{\alpha\rho}\delta^4(x_1 - y_1)\Lambda_\mu^{\mathbf{c}a}(x_i \mid x_2, y_2)_{\beta\sigma} \\
&+\mathcal{G}_{\mu\nu}^{ab}(x_i, y_1 \mid x_1, x_2; y_1, y_2)_{\alpha\beta\tau\sigma}(\Gamma^{a\mu})_{\tau\rho}
\end{aligned}
\tag{3.12}
$$

and

$$
\begin{aligned}
&\mathcal{G}_\mu^a(x_i \mid x_1, x_2; y_1, y_2)_{\alpha\beta\rho\delta}(i\overset{\leftarrow}{\partial}_{y_2} + m_2)_{\delta\sigma} = -\delta_{\beta\sigma}\delta^4(x_2 - y_2)\Lambda_\mu^a(x_i \mid x_1, y_1)_{\alpha\rho} \\
&+\mathcal{G}_{\mu\nu}^{ab}(x_i, y_2 \mid x_1, x_2; y_1, y_2)_{\alpha\beta\rho\delta}(\overline{\Gamma}^{b\nu})_{\delta\sigma}
\end{aligned}
\tag{3.13}
$$

where

$$
\begin{aligned}
&\mathcal{G}_{\mu\nu}^{ab}(x_i, y_j \mid x_1, x_2; y_1, y_2)_{\alpha\beta\rho\sigma} \\
&=\langle 0^+ \left| T\{N[\mathbf{A}_\mu^a(x_i)\psi_\alpha(x_1)\psi_\beta^c(x_2)]N[\mathbf{A}_\nu^b(y_j)\overline{\psi}_\rho(y_1)\overline{\psi}_\sigma^c(y_2)]\} \right| 0^- \rangle \\
&= G_{\mu\nu}^{ab}(x_i, y_j \mid x_1, x_2; y_1, y_2)_{\alpha\beta\rho\sigma} + \Lambda_\mu^{a*}(x_i \mid x_1, x_2)_{\alpha\beta}\overline{\Lambda}_\nu^{b*}(y_j \mid y_1, y_2)_{\rho\sigma}
\end{aligned}
\tag{3.14}
$$

Now, let us multiply Eqs. (3.5) and (3.6) respectively with γ_1^0 and γ_2^0 from the right and sum up the both equations thus obtained. In this way, writing in the matrix from, we obtain the following equation

$$
\begin{aligned}
&\mathcal{G}(x_1, x_2; y_1, y_2)[i\frac{\overset{\leftarrow}{\partial}}{\partial t'} + \overleftarrow{H}_0(y_1, y_2)] \\
&= -S(x_1, x_2; y_1, y_2) - \sum_{i=1}^2 \mathcal{G}^{(i)}(y_i \mid x_1, x_2; y_1, y_2)
\end{aligned}
\tag{3.15}
$$

where

$$
\frac{\partial}{\partial t'} = \frac{\partial}{\partial y_1^0} + \frac{\partial}{\partial y_2^0} = \frac{\partial}{\partial Y^0}, \tag{3.16}
$$

$$
H_0(y_1, y_2) = h^{(1)}(y_1) + h^{(2)}(y_2) \tag{3.17}
$$

here $h^{(1)}(y_1)$ and $h^{(2)}(y_2)$ were represented in Eq. (2.21),

$$
\begin{aligned}
\mathcal{G}^{(1)}(y_1 \mid x_1, x_2; y_1, y_2)_{\alpha\beta\rho\sigma} &= \mathcal{G}_\mu^a(y_1 \mid x_1, x_2; y_1, y_2)_{\alpha\beta\tau\sigma}(\overline{\Omega}_1^{a\mu})_{\tau\rho} \\
\mathcal{G}^{(2)}(y_2 \mid x_1, x_2; y_1, y_2)_{\alpha\beta\rho\sigma} &= \mathcal{G}_\mu^a(y_2 \mid x_1, x_2; y_1, y_2)_{\alpha\beta\rho\delta}(\overline{\Omega}_2^{a\mu})_{\delta\sigma}
\end{aligned}
\tag{3.18}
$$

in which

$$
\overline{\Omega}_1^{a\mu} = -g\gamma_1^\mu\gamma_1^0 T^a = -g\gamma_1^0\gamma_1^{\mu+}T^a, \quad \overline{\Omega}_2^{a\mu} = -g\gamma_2^\mu\gamma_2^0\overline{T}^a = --g\gamma_2^0\gamma_2^{\mu+}\overline{T}^a \tag{3.19}
$$

and $S(x_1, x_2; y_1, y_2)$ was defined in Eq. (2.29).

Similarly, on multiplying Eqs. (3.12) and (3.13) respectively with γ_1^0 and γ_2^0 from the right and summing up the both equations thus obtained, one can get

$$\mathcal{G}_\mu^a(x_i \mid x_1, x_2; y_1, y_2)_{\alpha\beta\lambda\tau}[i\overleftarrow{\frac{\partial}{\partial t'}} + \overleftarrow{H_0}(y_1, y_2)]_{\lambda\tau\rho\sigma}$$
$$= -\mathcal{R}_\mu^{(i)a}(x_1, x_2; y_1, y_2)_{\alpha\beta\rho\sigma} - \mathcal{Q}_\mu^{(i)a}(x_1, x_2; y_1, y_2)_{\alpha\beta\rho\sigma} \tag{3.20}$$

where

$$\mathcal{R}_\mu^{(i)a}(x_1, x_2; y_1, y_2)_{\alpha\beta\rho\sigma} = \delta^4(x_1 - y_1)(\gamma_1^0)_{\alpha\rho}\Lambda_\mu^{ca}(x_i \mid x_2, y_2)_{\beta\sigma}$$
$$+ \delta^4(x_2 - y_2)(\gamma_2^0)_{\beta\sigma}\Lambda_\mu^a(x_i \mid x_1, y_1)_{\alpha\rho} \tag{3.21}$$

and

$$\mathcal{Q}_\mu^{(i)a}(x_1, x_2; y_1, y_2)_{\alpha\beta\rho\sigma} = \mathcal{G}_{\mu\nu}^{ab}(x_i, y_1 \mid x_1, x_2; y_1, y_2)_{\alpha\beta\tau\sigma}(\overline{\Omega}_1^{b\nu})_{\tau\rho}$$
$$+ \mathcal{G}_{\mu\nu}^{ab}(x_i, y_2 \mid x_1, x_2; y_1, y_2)_{\alpha\beta\rho\delta}(\overline{\Omega}_2^{b\nu})_{\delta\sigma} \tag{3.22}$$

With the definitions given in Eq. (2.22) and in the following

$$\mathcal{R}^{(i)}(x_1, x_2; y_1, y_2) = \Omega_i^{a\mu}\mathcal{R}_\mu^{(i)a}(x_1, x_2; y_1, y_2) \tag{3.23}$$

$$\mathcal{Q}^{(i)}(x_1, x_2; y_1, y_2) = \Omega_i^{a\mu}\mathcal{Q}_\mu^{(i)a}(x_1, x_2; y_1, y_2) \tag{3.24}$$

we can write from Eq. (3.20) the equation satisfied by the function $\mathcal{G}_{\alpha\beta\rho\sigma}^{(i)}(x_i \mid x_1, x_2; y_1, y_2)$. In the matrix form, it reads

$$\mathcal{G}^{(i)}(x_i \mid x_1, x_2; y_1, y_2)[i\overleftarrow{\frac{\partial}{\partial t'}} + \overleftarrow{H_0}(y_1, y_2)]$$
$$= -\mathcal{R}^{(i)}(x_1, x_2; y_1, y_2) - \mathcal{Q}^{(i)}(x_1, x_2; y_1, y_2) \tag{3.25}$$

Up to the present, we are ready to derive the interaction kernel. Acting on the both sides of Eq. (2.24) with the operator $i\overleftarrow{\frac{\partial}{\partial t'}} + \overleftarrow{H_0}(y_1, y_2)$ from the right and employing Eqs. (3.15) and (3.25), we have

$$\int d^4z_1 d^4z_2 K^{(i)}(x_1, x_2; z_1, z_2)S(z_1, z_2; y_1, y_2) = \mathcal{R}^{(i)}(x_1, x_2; y_1, y_2) + \mathcal{Q}^{(i)}(x_1, x_2; y_1, y_2)$$
$$- \int d^4z_1 d^4z_2 K^{(i)}(x_1, x_2; z_1, z_2) \sum_{j=1}^2 \mathcal{G}^{(j)}(y_j \mid z_1, z_2; y_1, y_2) \tag{3.26}$$

In order to obtain the kernel, we may operate on the above equation with the inverse of $S(x_1, x_2; y_1, y_2)$ and the kernel on the RHS of Eq. (3.26) may be eliminated by the following relation given by acting on Eq. (2.24) with the inverse of the Green's function $\mathcal{G}(x_1, x_2; y_1, y_2)$

$$K^{(i)}(x_1, x_2; y_1, y_2) = \int d^4u_1 d^4u_2 \mathcal{G}^{(i)}(x_i \mid x_1, x_2; u_1, u_2)\mathcal{G}^{-1}(u_1, u_2; y_1, y_2) \tag{3.27}$$

With these operations, one may derive from Eq.(3.26) a closed expression of the kernel $K^{(i)}(x_1, x_2; y_1, y_2)$ such that

$$K^{(i)}(x_1, x_2; y_1, y_2) = \int d^4z_1 d^4z_2 \{\mathcal{R}^{(i)}(x_1, x_2; z_1, z_2) + \mathcal{Q}^{(i)}(x_1, x_2; z_1, z_2)$$
$$- \mathcal{D}^{(i)}(x_1, x_2; z_1, z_2)\}S^{-1}(z_1, z_2; y_1, y_2) \tag{3.28}$$

where

$$\mathcal{D}^{(i)}(x_1, x_2; z_1, z_2)$$
$$= \int \prod_{k=1}^2 d^4u_k d^4v_k \mathcal{G}^{(i)}(x_i \mid x_1, x_2; u_1, u_2)\mathcal{G}^{-1}(u_1, u_2; v_1, v_2) \sum_{j=1}^2 \mathcal{G}^{(j)}(y_j \mid v_1, v_2; z_1, z_2) \tag{3.29}$$

In the above derivation, existence of the inverses $\mathcal{G}^{-1}(u_1, u_2; z_1, z_2)$ and $S^{-1}(z_1, z_2; y_1, y_2)$ has been assumed. The rationality of the assumption will be illustrated later.

Clearly, the total interaction kernel appearing in Eq. (2.38) is given by the sum

$$
\begin{aligned}
K(x_1, x_2; y_1, y_2) &= \sum_{j=1}^{2} K^{(i)}(x_1, x_2; y_1, y_2) = \int d^4 z_1 d^4 z_2 \{ \mathcal{R}(x_1, x_2; z_1, z_2) \\
&+ \mathcal{Q}(x_1, x_2; z_1, z_2) - \mathcal{D}(x_1, x_2; z_1, z_2) \} S^{-1}(z_1, z_2; y_1, y_2)
\end{aligned} \tag{3.30}
$$

where

$$
\mathcal{R}(x_1, x_2; z_1, z_2) = \sum_{i=1}^{2} \mathcal{R}^{(i)}(x_1, x_2; z_1, z_2) \tag{3.31}
$$

$$
\mathcal{Q}(x_1, x_2; z_1, z_2) = \sum_{i=1}^{2} \mathcal{Q}^{(i)}(x_1, x_2; z_1, z_2) = \sum_{i,j=1}^{2} \Omega_i^{a\mu} \mathcal{G}_{\mu\nu}^{ab}(x_i, z_j \mid x_1, x_2; z_1, z_2) \overline{\Omega}_j^{b\nu} \tag{3.32}
$$

and

$$
\begin{aligned}
&\mathcal{D}(x_1, x_2; z_1, z_2) \\
&= \int \prod_{k=1}^{2} d^4 u_k d^4 v_k \sum_{i,j=1}^{2} \mathcal{G}^{(i)}(x_i \mid x_1, x_2; u_1, u_2) \mathcal{G}^{-1}(u_1, u_2; v_1, v_2) \mathcal{G}^{(j)}(y_j \mid v_1, v_2; z_1, z_2)
\end{aligned} \tag{3.33}
$$

We would like here to discuss the role played by the last term in Eq. (3.30). In view of the relation in Eq. (2.24) and the following relation

$$
\mathcal{G}^{(j)}(y_j \mid x_1, x_2; y_1, y_2) = \int d^4 z_1 d^4 z_2 \mathcal{G}(x_1, x_2; z_1, z_2) K^{(j)}(z_1, z_2; y_1, y_2) \tag{3.34}
$$

which also follows from the B-S reducibility of the Green's function $\mathcal{G}^{(j)}(y_j \mid x_1, x_2; y_1, y_2)$ and considering

$$
\int d^4 z_1 d^4 z_2 \mathcal{G}(x_1, x_2; z_1, z_2) \mathcal{G}^{-1}(z_1, z_2; y_1, y_2) = \delta^4(x_1 - y_1) \delta^4(x_2 - y_2) \tag{3.35}
$$

the function $\mathcal{D}(x_1, x_2; z_1, z_2)$ in Eq. (3.33) may be represented as

$$
\mathcal{D}(x_1, x_2; z_1, z_2) = \int \prod_{k=1}^{2} d^4 u_k d^4 v_k K(x_1, x_2; u_1, u_2) \mathcal{G}(u_1, u_2; v_1, v_2) K(v_1, v_2; z_1, z_2) \tag{3.36}
$$

which manifests itself the typical structure of the B-S reducibility of the kernel. Therefore, the last term in Eq. (3.30) just plays the role of cancelling the B-S reducible part of the remaining terms in Eq. (3.30). As a result of the cancellation, the interaction kernel given in Eq. (3.30) is really B-S irreducible, consistent with the ordinary concept for the kernel in a two-body relativistic equation. Inserting Eq. (3.36) into Eq. (3.30), writing in the operator form, we have

$$
KS = \mathcal{R} + \mathcal{Q} - K\mathcal{G}K \tag{3.37}
$$

This can be regarded as the integral equation satisfied by the kernel K.

Analogously, In accord with the definition in Eq. (2.35) and the expression in Eq. (3.28), one may write out an explicit expression of the kernel occurring in Eq. (2.39)

$$
\begin{aligned}
\overline{K}(x_1, x_2; y_1, y_2) &= \eta_2 K^{(1)}(x_1, x_2; y_1, y_2) - \eta_1 K^{(2)}(x_1, x_2; y_1, y_2) \\
&= \int d^4 z_1 d^4 z_2 \{ \overline{\mathcal{R}}(x_1, x_2; z_1, z_2) + \overline{\mathcal{Q}}(x_1, x_2; z_1, z_2) \\
&- \overline{\mathcal{D}}(x_1, x_2; z_1, z_2) \} S^{-1}(z_1, z_2; y_1, y_2)
\end{aligned} \tag{3.38}
$$

in which

$$\overline{\mathcal{A}}(x_1, x_2; z_1, z_2) = \eta_2 \mathcal{A}^{(1)}(x_1, x_2; z_1, z_2) - \eta_1 \mathcal{A}^{(2)}(x_1, x_2; z_1, z_2) \tag{3.39}$$

where $\mathcal{A}$ stands for $\mathcal{R}, \mathcal{Q}$ or $\mathcal{D}$.

Now, we are in a position to write out the interaction kernel appearing in the three-dimensional D-S equation. As demonstrated in Ref. [15], the kernel in Eq. (2.54) can be derived by the same procedure as for the four-dimensional kernel. The expression of the three-dimensional kernel shown below formally is the same as the four-dimensional one described in Eq. (3.30) except that it is now represented in the three-dimensional space.

$$K(\overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{y_1}, \overrightarrow{y_2}; E) = \int d^3 z_1 d^3 z_2 \{ \mathcal{R}(\overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}) \\ + \mathcal{Q}(\overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}; E) - \mathcal{D}(\overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}; E) \} S^{-1}(\overrightarrow{z_1}, \overrightarrow{z_2}; \overrightarrow{y}_1, \overrightarrow{y}_2) \tag{3.40}$$

where $\mathcal{R}(\overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}), \mathcal{Q}(\overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}; E)$ and $\mathcal{D}(\overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}; E)$ can be written out from Eqs. (3.31)-(3.33) as follows:

$$\mathcal{R}(\overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}) = \sum_{i=1}^{2} \Omega_i^{a\mu} \mathcal{R}_\mu^{(i)a}(\overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}) \tag{3.41}$$

in which

$$\mathcal{R}_\mu^{(i)a}(\overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}) = \delta^3(\overrightarrow{x_1} - \overrightarrow{z_1}) \gamma_1^0 \Lambda_\mu^{\mathbf{c}a}(\overrightarrow{x_i} \mid \overrightarrow{x_2}, \overrightarrow{z_2}) + \delta^3(\overrightarrow{x_2} - \overrightarrow{z_2}) \gamma_2^0 \Lambda_\mu^a(\overrightarrow{x_i} \mid \overrightarrow{x_1}, \overrightarrow{z_1}) \tag{3.42}$$

here $\Lambda_\mu^a(\overrightarrow{x_i} \mid \overrightarrow{x_1}, \overrightarrow{y_1})$ and $\Lambda_\mu^{\mathbf{c}a}(\overrightarrow{x_i} \mid \overrightarrow{x_2}, \overrightarrow{y_2})$ are defined as in Eq. (3.10) except that the time variables in all the field operators are now taken to be the same and therefore they are time-independent due to the translation-invariance property of the Green's functions,

$$\mathcal{Q}(\overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}; E) = \sum_{i,j=1}^{2} \Omega_i^{a\mu} \mathcal{G}_{\mu\nu}^{ab}(\overrightarrow{x_i}, \overrightarrow{z_j} \mid \overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}; E) \overline{\Omega}_j^{b\nu} \tag{3.43}$$

in which $\mathcal{G}_{\mu\nu}^{ab}(\overrightarrow{x_i}, \overrightarrow{z_j} \mid \overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{z}_1, \overrightarrow{z}_2; E)$ is the Fourier transform of the Green's function defined by

$$\mathcal{G}_{\mu\nu}^{ab}(\overrightarrow{x_i}, \overrightarrow{z_j} \mid \overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{z}_1, \overrightarrow{z}_2; t_1 - t_2) \\ = \langle 0^+ \left| T\{ N[\mathbf{A}_\mu^a(\overrightarrow{x_i}, t_1) \psi(\overrightarrow{x_1}, t_1) \psi^c(\overrightarrow{x_2}, t_1)] N[\mathbf{A}_\nu^b(\overrightarrow{z_j}, t_2) \overline{\psi}(\overrightarrow{z_1}, t_2) \overline{\psi}^c(\overrightarrow{z_2}, t_2)] \} \right| 0^- \rangle \tag{3.44}$$

and

$$\mathcal{D}(\overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}; E) = \int \prod_{k=1}^{2} d^3 u_k d^3 v_k \sum_{i,j=1}^{2} \Omega_i^{a\mu} \mathcal{G}_\mu^{(i)a}(\overrightarrow{x_i} \mid \overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{u_1}, \overrightarrow{u_2}; E) \\ \times \mathcal{G}^{-1}(\overrightarrow{u_1}, \overrightarrow{u_2}; \overrightarrow{v}_1, \overrightarrow{v}_2; E) \mathcal{G}_\nu^{(j)b}(\overrightarrow{z_j} \mid \overrightarrow{v}_1, \overrightarrow{v}_2; \overrightarrow{z_1}, \overrightarrow{z_2}; E) \overline{\Omega}_j^{b\nu} \tag{3.45}$$

in which $\mathcal{G}_\mu^{(i)a}(\overrightarrow{x_i} \mid \overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{u_1}, \overrightarrow{u_2}; E)$ and $\mathcal{G}_\nu^{(j)b}(\overrightarrow{z_j} \mid \overrightarrow{v}_1, \overrightarrow{v}_2; \overrightarrow{z_1}, \overrightarrow{z_2}; E)$ are the Fourier transforms of the following Green's functions

$$\mathcal{G}_\mu^{(i)a}(\overrightarrow{x_i} \mid \overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{u_1}, \overrightarrow{u_2}; t_1 - t_2) \\ = \langle 0^+ \left| T\{ N[\mathbf{A}_\mu^a(\overrightarrow{x_i}, t_1) \psi(\overrightarrow{x_1}, t_1) \psi^c(\overrightarrow{x_2}, t_1)] N[\overline{\psi}(\overrightarrow{u_1}, t_2) \overline{\psi}^c(\overrightarrow{u_2}, t_2)] \} \right| 0^- \rangle \tag{3.46}$$

and

$$\mathcal{G}_\nu^{(j)b}(\overrightarrow{z_j} \mid \overrightarrow{v}_1, \overrightarrow{v}_2; \overrightarrow{z_1}, \overrightarrow{z_2}; t_1 - t_2) \\ = \langle 0^+ \left| T\{ N[\psi(\overrightarrow{v_1}, t_1) \psi^c(\overrightarrow{v_2}, t_1)] N[\mathbf{A}_\nu^b(\overrightarrow{z_j}, t_2) \overline{\psi}(\overrightarrow{z_1}, t_2) \overline{\psi}^c(\overrightarrow{z_2}, t_2)] \} \right| 0^- \rangle \tag{3.47}$$

and $S^{-1}(\overrightarrow{z_1}, \overrightarrow{z_2}; \overrightarrow{y_1}, \overrightarrow{y_2})$ is the inverse of the function in Eq. (2.50).

4. Another derivation of the interaction kernel

The aim of this section is to give a different expression of the interaction kernel in Eqs. (2.38) and (2.39) which will be derived by means of the irreducible decomposition of the Green's functions $\mathcal{G}_\mu^a(x_i \mid x_1, x_2; y_1, y_2)$ denoted in Eq. (2.18). First, we start from the relation between the full $q\bar{q}$ four-point Green's function $G(x_1, x_2; y_1, y_2)$ and its connected one $G_c(x_1, x_2; y_1, y_2)$ which is derived in the beginning of Appendix B [18, 33]

$$G(x_1, x_2; y_1, y_2) = G_c(x_1, x_2; y_1, y_2) + S_F(x_1 - y_1) S_F^c(x_2 - y_2) \\ -S_F^*(x_1 - x_2) \overline{S}_F^*(y_1 - y_2) \tag{4.1}$$

where all the fermion propagators were defined in Eqs. (2.6)-(2.9). Substituting Eq. (4.1) into Eq. (2.4), we get

$$\mathcal{G}(x_1, x_2; y_1, y_2) = G_c(x_1, x_2; y_1, y_2) + S_F(x_1 - y_1) S_F^c(x_2 - y_2) \tag{4.2}$$

where the last term is the unconnected part of the function $\mathcal{G}(x_1, x_2; y_1, y_2)$.

From Eq. (B.5) given in Appendix B, we obtain by setting the source $J = 0$ that [18, 33]

$$G_\mu^a(x_i \mid x_1, x_2; y_1, y_2) = G_{c\mu}^a(x_i \mid x_1, x_2; y_1, y_2) + \Lambda_\mu^a(x_i \mid x_1; y_1) S_F^c(x_1 - y_1) \\ +S_F(x_1 - y_1) \Lambda_\mu^{\mathbf{c}a}(x_i \mid x_2; y_2) - \Lambda_\mu^{a*}(x_i \mid x_1, x_2) \overline{S}_F^*(y_1 - y_2) - S_F^*(x_1 - x_2) \overline{\Lambda}_\mu^{a*}(x_i \mid y_1, y_2) \tag{4.3}$$

where $i = 1, 2$, $G_{c\mu}^a(x_i \mid x_1, x_2; y_1, y_2)$ is the connected part of the Green's function $G_\mu^a(x_i \mid x_1, x_2; y_1, y_2)$ and $\Lambda_\mu^{a*}(x_i \mid x_1, x_2)$, $\Lambda_\mu^a(x_i \mid x_1; y_1)$, $\Lambda_\mu^{\mathbf{c}a}(x_i \mid x_2; y_2)$ and $\overline{\Lambda}_\mu^{a*}(x_i \mid y_1, y_2)$ are the three-point Green's functions which are given in Eqs. (2.15) and (3.10). On inserting the decomposition in Eq. (4.3) into Eq. (2.18), we see that the last unconnected term in Eq. (2.18) is cancelled out. Thus, we have

$$\mathcal{G}_\mu^a(x_i \mid x_1, x_2; y_1, y_2) = G_{c\mu}^a(x_i \mid x_1, x_2; y_1, y_2) + \Lambda_\mu^a(x_i \mid x_1; y_1) S_F^c(x_2 - y_2) \\ +S_F(x_1 - y_1) \Lambda_\mu^{\mathbf{c}a}(x_i \mid x_2; y_2) - S_F^*(x_1 - x_2) \overline{\Lambda}_\mu^{a*}(x_i \mid y_1, y_2) \tag{4.4}$$

where $i = 1, 2$. Substituting the above expression in Eq. (2.22), we will obtain the decomposition of the function $\mathcal{G}^{(i)}(x_i \mid x_1, x_2; y_1, y_2)$.

In the following, we are devoted to analyzing the terms on the RHS of Eq. (4.4) through the one-particle-irreducible decompositions of the connected Green's functions included in those terms. The decompositions have been carried out in Appendix B.

A. The t-channel one-gluon exchange kernel

First we focus our attention on the second and third terms in Eq. (4.4). According to the decomposition in Eq. (B.15), the three-point gluon-quark Green's functions $\Lambda_\mu^a(x_i \mid x_j; y_k)$ which is fully connected can be represented in the form

$$\Lambda_\mu^a(x_i \mid x_j; y_k) = \int d^4 z_1 \Sigma_\mu^a(x_i \mid x_j; z_1) S_F(z_1 - y_k) \tag{4.5}$$

where

$$\Sigma_\mu^a(x_i \mid x_j; z_1) = \int d^4 u_1 d^4 u_2 \Delta_{\mu\nu}^{ab}(x_i - u_1) S_F(x_j - u_2) \Gamma^{b\nu}(u_1 \mid u_2, z_1) \tag{4.6}$$

in which

$$\Delta_{\mu\nu}^{ab}(x_i - u_j) = \frac{1}{i} \left\langle 0^+ \left| T[\mathbf{A}_\mu^a(x_i) \mathbf{A}_\nu^b(u_j)] \right| 0^- \right\rangle \tag{4.7}$$

is the exact gluon propagator and $\Gamma^{b\nu}(u_1 \mid u_2, z_1)$ is the gluon-quark three-line proper vertex as defined in Eq. (B.17). The one-particle-irreducible decompositions of the three-point gluon-antiquark Green's functions $\Lambda_\mu^{\mathbf{c}a}(x_i \mid x_j; y_k)$ can be obtained from Eqs. (4.5) and (4.6) by the charge conjugation transformation. The result is

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$$\Lambda_\mu^{\mathbf{c}a}(x_i \mid x_j; y_k) = \int d^4 z_2 \Sigma_\mu^{\mathbf{c}a}(x_i \mid x_j; z_2) S_F^c(z_2 - y_k) \tag{4.8}$$

where

$$\Sigma_\mu^{ca}(x_i \mid x_j; z_2) = \int d^4 u_1 d^4 u_2 \Delta_{\mu\nu}^{ab}(x_i - u_1) S_F^c(x_j - u_2) \Gamma_c^{b\nu}(u_1 \mid u_2, z_2) \tag{4.9}$$

in which $\Gamma_c^{b\nu}(u_1 \mid u_2, z_2)$ is the gluon-antiquark three-line proper vertex as defined in Eq. (B.18).

When we set $i = j$ in Eqs. (4.5) and (4.8), it is found that the $\Sigma_\mu^a(x_1 \mid x_1; z_1)$ in Eq. (4.6) and the $\Sigma_\mu^{\mathbf{c}a}(x_2 \mid x_2; z_2)$ in Eq. (4.9) are respectively related to the quark and antiquark self-energies in such a way

$$\Omega_1^{a\mu}\Sigma_\mu^a(x_1 \mid x_1; z_1) = -\gamma_1^0 \Sigma(x_1, z_1) = \widehat{\Sigma}(x_1, z_1),$$
$$\Omega_2^{a\mu}\Sigma_\mu^{\mathbf{c}a}(x_2 \mid x_2; z_2) = -\gamma_2^0 \Sigma^c(x_2, z_2) = \widehat{\Sigma}^c(x_2, z_2) \tag{4.10}$$

Thus, from Eqs. (4.5), (4.8) and (4.10), we have

$$\Omega_1^{a\mu}\Lambda_\mu^a(x_1 \mid x_1; y_1) S_F^c(x_2 - y_2) + \Omega_2^{a\mu}\Lambda_\mu^{\mathbf{c}a}(x_2 \mid x_2; z) S_F(x_1 - y_1)$$
$$= \int d^4 z_1 d^4 z_2 \Sigma(x_1, x_2; z_1, z_2) S_F(z_1 - y_1) S_F^c(z_2 - y_2) \tag{4.11}$$

where

$$\Sigma(x_1, x_2; z_1, z_2) = \widehat{\Sigma}(x_1, z_1)\delta^4(x_2 - z_2) + \widehat{\Sigma}^c(x_2, z_2)\delta^4(x_1 - z_1) \tag{4.12}$$

which is the total self-energy of the $q\bar{q}$ system. According to the definitions given in Eqs. (4.4), (2.22)-(2.24), (2.31) and (4.2), we see, $\Sigma(x_1, x_2; z_1, z_2)$ as a self energy term to appear in the interaction kernel.

In the case of $i \neq j$, from Eqs. (4.5) and (4.8), it can be written

$$\Omega_2^{a\mu}\Lambda_\mu^a(x_2 \mid x_1; y_1) S_F^c(x_2 - y_2) + \Omega_1^{a\mu}\Lambda_\mu^{\mathbf{c}a}(x_1 \mid x_2; y_2) S_F(x_1 - y_1)$$
$$= \int d^4 z_1 d^4 z_2 K_t(x_1, x_2; z_1, z_2) S_F(z_1 - y_1) S_F^c(z_2 - y_2) \tag{4.13}$$

where

$$K_t(x_1, x_2; z_1, z_2) = \Omega_2^{a\mu}\Sigma_\mu^a(x_2 \mid x_1; z_1)\delta^4(x_2 - z_2) + \Omega_1^{a\mu}\Sigma_\mu^{\mathbf{c}a}(x_1 \mid x_2; z_2)\delta^4(x_1 - z_1) \tag{4.14}$$

Based on Eqs. (4.4), (2.22)-(2.24), (2.31) and (4.2), it is clear that the $K_t(x_1, x_2; z_1, z_2)$ acts as a part of the interaction kernel to appear in the D-S equation. As will be seen in section 6, this part is precisely the kernel of t-channel one-gluon exchange interaction..

B. The s-channel one-gluon exchange kernel

Next, we turn to the last term in Eq. (4.4). The one-particle irreducible decomposition of the three-point Green's function in this term may also be found from Eqs. (4.5) and (4.6) by the charge conjugation transformation. The result is

$$\overline{\Lambda}_\mu^{a*}(x_i \mid y_1, y_2) = -\int d^4 z_1 d^4 z_2 d^4 z \Delta_{\mu\nu}^{ab}(x_i - z)\overline{\Gamma}^{b\nu*}(z \mid z_1, z_2) S_F(z_1 - y_1) S_F^c(z_2 - y_2) \tag{4.15}$$

where $\overline{\Gamma}^{b\nu*}(z \mid z_1, z_2)$ is the gluon-quark-antiquark proper vertex defined in Eq. (B.19). With the decomposition given above, according to Eqs. (2.22)-(2.24) and (2.31), the contribution of the last term in Eq. (4.4) to the kernel $K(x_1, x_2; y_1, y_2)$ can be found from the sum

$$-\sum_{i=1}^{2} \Omega_i^{a\mu} S_F^*(x_1 - x_2)\overline{\Lambda}_\mu^{a*}(x_i \mid y_1, y_2) = \int d^4 z_1 d^4 z_2 K_s(x_1, x_2; z_1, z_2) S_F(z_1 - y_1) S_F^c(z_2 - y_2) \tag{4.16}$$

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where

$$K_s(x_1, x_2; z_1, z_2) = S_F^*(x_1 - x_2) \int d^4z \sum_{i=1}^{2} \Omega_i^{a\mu} \Delta_{\mu\nu}^{ab}(x_i - z) \overline{\Gamma}^{b\nu*}(z \mid z_1, z_2) \tag{4.17}$$

is just the s-channel one-gluon exchange kernel occurring in the D-S equation which will be discussed in section 6. It is noted here that the s-channel one-gluon exchange describes the $q\overline{q}$ annihilation and creation process which takes place between the quark (antiquark) in the initial state and the antiquark (quark) in the final state as indicated by the gluon propagator in Eq. (4.17) (not between the quark and the antiquark both of which are simultaneously related to the initial state or the final state B-S amplitude for a bound state).

C. The kernel from the Green's function $G_{c\mu}^a(x_i \mid x_1, x_2; y_1, y_2)$

Now let us concentrate our attention on the irreducible decomposition of the first term in Eq. (4.4). As stated in Appendix B, this decomposition may be derived from the functional differential of the Green's function $G_c(x_1, x_2; y_1, y_2)$ with respect to the source $J^{a\mu}(x_i)$ by using the one-particle irreducible decomposition of the function $G_c(x_1, x_2; y_1, y_2)$. The latter decomposition whose derivation is sketched in Appendix B is well-known [18, 33] and can be represented in the form

$$\begin{aligned} G_c(x_1, x_2; y_1, y_2) = & \int \prod_{i=1}^{2} d^4u_i d^4v_i S_F(x_1 - u_1) S_F^c(x_2 - u_2) \\ & \times \Gamma(u_1, u_2; v_1, v_2) S_F(v_1 - y_1) S_F^c(v_2 - y_2) \end{aligned} \tag{4.18}$$

where

$$\Gamma(u_1, u_2; v_1, v_2) = \sum_{i=1}^{3} \Gamma_i(u_1, u_2; v_1, v_2) \tag{4.19}$$

in which

$$\Gamma_1(u_1, u_2; v_1, v_2) = -\int d^4z_1 d^4z_2 \Gamma^{b\nu}(z_1 \mid u_1, v_1) D_{\nu\nu'}^{bb'}(z_1 - z_2) \Gamma_c^{b'\nu'}(z_2 \mid u_2, v_2) \tag{4.20}$$

$$\Gamma_2(u_1, u_2; v_1, v_2) = \int d^4z_1 d^4z_2 \Gamma^{*b\nu}(z_1 \mid u_1, u_2) D_{\nu\nu'}^{bb'}(z_1 - z_2) \overline{\Gamma}^{*b'\nu'}(z_2 \mid v_1, v_2) \tag{4.21}$$

here the three-line vertices are defined in Eqs. (B.17)-(B.19) and (B.21), $D_{\nu\nu'}^{bb'}(z_1 - z_2) = i\Delta_{\nu\nu'}^{bb'}(z_1 - z_2)$ with $\Delta_{\nu\nu'}^{bb'}(z_1 - z_2)$ defined in Eq. (4.7) and $\Gamma_3(u_1, u_2; v_1, v_2)$ defined in Eq. (B.22) is the quark-antiquark four-line proper vertex. After substituting the expressions in Eqs. (4.18)-(4.21), which are now given in the presence of source J, into Eq. (B.6) and completing the differentiation, the one-particle irreducible decomposition of the Green's function $G_{c\mu}^a(x_i \mid x_1, x_2; y_1, y_2)$ will be found and, thereby, we can write

$$\begin{aligned} \sum_{i=1}^{2} G_c^{(i)}(x_i \mid x_1, x_2; y_1, y_2) &\equiv \Omega_1^{a\mu} G_{c\mu}^a(x_1 \mid x_1, x_2; y_1, y_2) + \Omega_2^{a\mu} G_{c\mu}^a(x_2 \mid x_1, x_2; y_1, y_2) \\ &= \sum_{j=1}^{3} G_j(x_1, x_2; y_1, y_2) \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} & G_1(x_1, x_2; y_1, y_2) \\ &= \int \prod_{j=1}^{2} d^4u_j d^4v_j \sum_{i=1}^{2} \Omega_i^{a\mu} [\Lambda_\mu^a(x_i \mid x_1; u_1) S_F^c(x_2 - u_2) + S_F(x_1 - u_1) \\ & \times \Lambda_\mu^{ca}(x_i \mid x_2; u_2)] \Gamma(u_1, u_2; v_1, v_2) S_F(v_1 - y_1) S_F^c(v_2 - y_2) \end{aligned} \tag{4.23}$$

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$$G_2(x_1, x_2; y_1, y_2)$$
$$= \int \prod_{j=1}^{2} d^4 u_j d^4 v_j \sum_{i=1}^{2} \Omega_i^{a\mu} S_F(x_1 - u_1) S_F^c(x_2 - u_2) \Gamma(u_1, u_2; v_1, v_2) \tag{4.24}$$
$$\times [\Lambda_\mu^a(x_i \mid v_1; y_1) S_F^c(v_2 - y_2) + \Lambda_\mu^{ca}(x_i \mid v_2; y_2) S_F(v_1 - y_1)]$$

and

$$G_3(x_1, x_2; y_1, y_2)$$
$$= \int \prod_{j=1}^{2} d^4 u_j d^4 v_j \sum_{i=1}^{2} \Omega_i^{a\mu} S_F(x_1 - u_1) S_F^c(x_2 - u_2) \tag{4.25}$$
$$\times \Gamma^{a\mu}(x_i \mid u_1, u_2; v_1, v_2) S_F(v_1 - y_1) S_F^c(v_2 - y_2)$$

where $\Gamma^{a\mu}(x_i \mid u_1, u_2; v_1, v_2)$ is a kind of five-line vertex which is defined in Eq. (B.27) and will be specified soon later.

Before specifying the function $\Gamma^{a\mu}(x_i \mid u_1, u_2; v_1, v_2)$, we first analyze the expressions in Eqs. (4.23) and (4.24). According to the expressions in Eqs. (4.11), (4.13) and (4.18), Eq. (4.23) may be represented as

$$G_1(x_1, x_2; y_1, y_2) = \int d^4 z_1 d^4 z_2 [\Sigma(x_1, x_2; z_1, z_2) + K_t(x_1, x_2; z_1, z_2)] G_c(z_1, z_2; y_1, y_2) \tag{4.26}$$

where $\Sigma(x_1, x_2; z_1, z_2)$ and $K_t(x_1, x_2; z_1, z_2)$ were respectively described in Eqs. (4.12) and (4.14). In view of the decompositions in Eqs. (4.5) and (4.8), Eq. (4.24) may be written in the form

$$G_2(x_1, x_2; y_1, y_2) = \int d^4 z_1 d^4 z_2 K_1(x_1, x_2; z_1, z_2) S_F(z_1 - y_1) S_F^c(z_2 - y_2) \tag{4.27}$$

where

$$K_1(x_1, x_2; z_1, z_2) = \int \prod_{j=1}^{2} d^4 u_j d^4 v \sum_{i=1}^{2} \Omega_i^{a\mu} S_F(x_1 - u_1) S_F^c(x_2 - u_2) \tag{4.28}$$
$$\times [\Gamma(u_1, u_2; v, z_2) \Sigma_\mu^a(x_i \mid v; z_1) + \Gamma(u_1, u_2; z_1, v) \Sigma_\mu^{ca}(x_i \mid v; z_2)]$$

in which $\Sigma_\mu^a(x_i \mid v; z_1)$ and $\Sigma_\mu^{ca}(x_i \mid v; z_2)$ were represented in Eqs (4.6) and (4.9) respectively.

Let us turn to the five-line vertex function contained in Eq. (4.25). This vertex is two-particle reducible (or say, B-S reducible) although it is one-particle-irreducible. Therefore, it can be decomposed into a B-S irreducible part $\Gamma_{IR}^{a\mu}$ and a B-S reducible part $\Gamma_{RE}^{a\mu}$

$$\Gamma^{a\mu}(x_i \mid u_1, u_2; v_1, v_2) = \Gamma_{IR}^{a\mu}(x_i \mid u_1, u_2; v_1, v_2) + \Gamma_{RE}^{a\mu}(x_i \mid u_1, u_2; v_1, v_2) \tag{4.29}$$

In order to exhibit the above decomposition specifically, as mentioned in Appendix B, we may insert Eqs. (4.19)-(4.21), which are now given in the case of presence of the source J, into Eq. (B.27) and complete the differentiation with respect to the source $J^{a\mu}(x_i)$. After completing the differentiations shown in Eqs. (B.23) and (B.25), we can write

$$\Gamma^{a\mu}(x_i \mid u_1, u_2; v_1, v_2) = \sum_{j=1}^{3} \Gamma_j^{a\mu}(x_i \mid u_1, u_2; v_1, v_2) \tag{4.30}$$

where $\Gamma_j^{a\mu}(x_i \mid u_1, u_2; v_1, v_2)$ are given by the differential of the functions $\Gamma_j(u_1, u_2; v_1, v_2)$ in Eq. (4.19) and separately shown below

$$\Gamma_1^{a\mu}(x_i \mid u_1, u_2; v_1, v_2)$$
$$= -\int d^4 z d^4 z_1 d^4 z_2 D_{\mu\nu}^{ab}(x_i - z) \{\Gamma_\nu^{bc,\lambda}(z, z_1 \mid u_1, v_1) D_{\lambda\lambda'}^{cc'}(z_1 - z_2) \Gamma_{\mathbf{c}}^{c'\lambda'}(z_2 \mid u_2, v_2) \tag{4.31}$$
$$+ \Gamma^{c\lambda}(z_1 \mid u_1, v_1) [D_{\lambda\lambda'}^{cc'}(z_1 - z_2) \Gamma_{\mathbf{c}\nu}^{bc',\lambda}(z, z_2 \mid u_2, v_2) + \Pi_{\nu\lambda\lambda'}^{bcc'}(z, z_1, z_2) \Gamma^{c'\lambda'}(z_2 \mid u_2, v_2)]\}$$

in which besides the gluon-quark and gluon-antiquark three-line vertices mentioned before, there occur the gluon-quark four-line proper vertex $\Gamma_{\nu\lambda}^{bc}(z, z_1 \mid u_1, v_1)$ and the corresponding gluon-antiquark one $\Gamma_{\mathbf{c}\nu\lambda}^{bc}(z, z_1 \mid u_1, v_1)$ which are defined respectively in Eqs. (B.28) and (B.29) and

$$\Pi_{\nu\rho\sigma}^{bcd}(z, z_1, z_2) = \int d^4 u_1 d^4 u_2 D_{\rho\rho'}^{cc'}(z_1 - u_1)\Gamma_{bc'd'}^{\nu\rho'\sigma'}(z, u_1, u_2)D_{\sigma'\sigma}^{d'd}(u_2 - z_2) \tag{4.32}$$

in which $\Gamma_{bc'd'}^{\nu\rho'\sigma'}(z, u_1, u_2)$ is the gluon three-line proper vertex defined in Eq. (B.24),

$$\begin{aligned}
&\Gamma_2^{a\mu}(x_i \mid u_1, u_2; v_1, v_2) \\
&= \int d^4 z d^4 z_1 d^4 z_2 D_{\mu\nu}^{ab}(x_i - z)\{\Gamma_\nu^{bc,\lambda*}(z, z_1 \mid u_1, u_2)D_{\lambda\lambda'}^{cc'}(z_1 - z_2)\overline{\Gamma}_{\mathbf{c}}^{c'\lambda'*}(z_2 \mid v_1, v_2) \\
&+ \Gamma^{c\lambda*}(z_1 \mid u_1, u_2)[D_{\lambda\lambda'}^{cc'}(z_1 - z_2)\overline{\Gamma}_{\mathbf{c}\nu}^{bc',\lambda*}(z, z_2 \mid v_1, v_2) + \Pi_{\nu\lambda\lambda'}^{bcc'}(z, z_1, z_2)\ \overline{\Gamma}^{c'\lambda'*}(z_2 \mid v_1, v_2)]\}
\end{aligned} \tag{4.33}$$

in which $\Gamma_{\nu\lambda}^{bc*}(z, z_1 \mid u_1, u_2)$ and $\overline{\Gamma}_{\mathbf{c}\nu\lambda}^{bc*}(z, z_1 \mid v_1, v_2)$ are the gluon-quark-antiquark four-line proper vertices defined in Eqs. (B.30) and (B.31) and especially

$$\Gamma_3^{a\mu}(x_i \mid u_1, u_2; v_1, v_2) = \int d^4 z D_{\mu\nu}^{ab}(x_i - z)\widehat{\Gamma}^{b\nu}(z \mid u_1, u_2; v_1, v_2) \tag{4.34}$$

in which $\widehat{\Gamma}^{b\nu}(z \mid u_1, u_2; v_1, v_2)$ stands for the gluon-quark-antiquark five-line proper vertex defined in Eq. (B.33).

It is pointed out that the vertices formulated in Eqs. (4.31) and (4.33) are not only one-particle-irreducible, but also B-S irreducible. This point can be seen more clearly when the vertices are represented by their Feynman diagrams. From the diagrams, one can find that it is impossible to divide each of the diagrams into two unconnected parts by cutting two fermion lines. However, the five-line proper vertex $\widehat{\Gamma}^{b\nu}(z \mid u_1, u_2; v_1, v_2)$ in Eq. (4.34) is B-S reducible. It can be decomposed into a B-S reducible part $\widehat{\Gamma}_{RE}^{b\nu}$ and a B-S irreducible part $\widehat{\Gamma}_{IR}^{b\nu}$,

$$\widehat{\Gamma}^{b\nu}(z \mid u_1, u_2; v_1, v_2) = \widehat{\Gamma}_{IR}^{b\nu}(z \mid u_1, u_2; v_1, v_2) + \widehat{\Gamma}_{RE}^{b\nu}(z \mid u_1, u_2; v_1, v_2) \tag{4.35}$$

This enables us to write Eq. (4.34) as follows:

$$\Gamma_3^{a\mu}(x_i \mid u_1, u_2; v_1, v_2) = \Gamma_{31}^{a\mu}(x_i \mid u_1, u_2; v_1, v_2) + \Gamma_{32}^{a\mu}(x_i \mid u_1, u_2; v_1, v_2) \tag{4.36}$$

where

$$\Gamma_{31}^{a\mu}(x_i \mid u_1, u_2; v_1, v_2) = \int d^4 z D_{\mu\nu}^{ab}(x_i - z)\widehat{\Gamma}_{IR}^{b\nu}(z \mid u_1, u_2; v_1, v_2) \tag{4.37}$$

and

$$\Gamma_{32}^{a\mu}(x_i \mid u_1, u_2; v_1, v_2) = \int d^4 z D_{\mu\nu}^{ab}(x_i - z)\widehat{\Gamma}_{RE}^{b\nu}(z \mid u_1, u_2; v_1, v_2) \tag{4.38}$$

From the above statement, it is clearly seen that the B-S irreducible part of the vertex in Eq. (4.29) is given by the sum

$$\begin{aligned}
\Gamma_{IR}^{a\mu}(x_i \mid u_1, u_2; v_1, v_2) &= \Gamma_1^{a\mu}(x_i \mid u_1, u_2; v_1, v_2) \\
&+ \Gamma_2^{a\mu}(x_i \mid u_1, u_2; v_1, v_2) + \Gamma_{31}^{a\mu}(x_i \mid u_1, u_2; v_1, v_2)
\end{aligned} \tag{4.39}$$

where the three terms on the RHS of Eq. (4.39) were given in Eqs. (4.31), (4.33) and (4.37) respectively. While, the B-S reducible part in Eq. (4.29) is given by Eq. (4.38)

$$\Gamma_{RE}^{a\mu}(x_i \mid u_1, u_2; v_1, v_2) = \Gamma_{32}^{a\mu}(x_i \mid u_1, u_2; v_1, v_2) \tag{4.40}$$

Based on the decomposition in Eq. (4.29), the function in Eq. (4.25) will be decomposed into

$$G_3(x_1, x_2; y_1, y_2) = G_3^{(1)}(x_1, x_2; y_1, y_2) + G_3^{(2)}(x_1, x_2; y_1, y_2) \tag{4.41}$$

where

$$G_3^{(1)}(x_1, x_2; y_1, y_2) = \int d^4 z_1 d^4 z_2 K_2(x_1, x_2; z_1, z_2)S_F(z_1 - y_1)S_F^c(z_2 - y_2) \tag{4.42}$$

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in which

$$K_2(x_1, x_2; z_1, z_2) = \int \prod_{j=1}^{2} d^4 u_j \sum_{i=1}^{2} \Omega_i^{a\mu} S_F(x_1 - u_1) S_F^c(x_2 - u_2) \Gamma_{IR}^{a\mu}(x_i \mid u_1, u_2; z_1, z_2) \qquad (4.43)$$

and

$$\begin{aligned}G_3^{(2)}(x_1, x_2; y_1, y_2) = & \int \prod_{j=1}^{2} d^4 u_j d^4 v_j \sum_{i=1}^{2} \Omega_i^{a\mu} S_F(x_1 - u_1) S_F^c(x_2 - u_2) \\ & \times \Gamma_{RE}^{a\mu}(x_i \mid u_1, u_2; v_1, v_2) S_F(v_1 - y_1) S_F^c(v_2 - y_2)\end{aligned} \qquad (4.44)$$

It is emphasized that due to the B-S irreducibility of the vertex $\Gamma_{IR}^{a\mu}(x_i \mid u_1, u_2; v_1, v_2)$, the function $G_3^{(1)}(x_1, x_2; y_1, y_2)$ can only be written in the form as shown in Eq. (4.42). While, since the vertex $\Gamma_{RE}^{a\mu}(x_i \mid u_1, u_2; v_1, v_2)$ is B-S reducible, as was similarly done for the Green's function $\mathcal{G}^{(i)}(x_1, x_2; y_1, y_2)$, the function $G_3^{(2)}(x_1, x_2; y_1, y_2)$, as a part of the connected Green's function, must be represented in the B-S reducible form

$$G_3^{(2)}(x_1, x_2; y_1, y_2) = \int d^4 z_1 d^4 z_2 \widetilde{K}(x_1, x_2; z_1, z_2) G_c(z_1, z_2; y_1, y_2) \qquad (4.45)$$

where $\widetilde{K}(x_1, x_2; z_1, z_2)$ is a kind of kernel needed to be determined later.

D. Complete expression of the interaction kernel

Up to the present, the irreducible decompositions of the functions $\mathcal{G}^{(i)}(x_1, x_2; y_1, y_2)$ $(i = 1, 2)$ appearing in Eq. (2.24) have been completed. Now, let us first sum up the expressions given in Eqs. (4.11) and (4.13) which correspond to the second and third terms in Eq. (4.4) and the expression in Eq. (4.26) for the function $G_1(x_1, x_2; y_1, y_2)$ which is contained in the connected Green's functions $G_{c\mu}^a(x_i \mid x_1, x_2; y_1, y_2)$. The summation yields

$$\int d^4 z_1 d^4 z_2 [\Sigma(x_1, x_2; z_1, z_2) + K_t(x_1, x_2; z_1, z_2)] \mathcal{G}(z_1, z_2; y_1, y_2) \qquad (4.46)$$

where the relation in Eq. (4.2) has been considered. Then, we combine the expression in Eq. (4.16) which corresponds to the last term in Eq. (4.4) and the expressions in Eqs. (4.27) and (4.42) which come from the B-S irreducible part of connected Green's functions $G_{c\mu}^a(x_i \mid x_1, x_2; y_1, y_2)$ and obtain

$$\int d^4 z_1 d^4 z_2 [K_s(x_1, x_2; z_1, z_2) + K_1(x_1, x_2; z_1, z_2) + K_2(x_1, x_2; z_1, z_2)] S_F(z_1 - y_1) S_F^c(z_2 - y_2) \qquad (4.47)$$

The final decomposition of the sum of the functions $\mathcal{G}^{(1)}(x_1 \mid x_1, x_2; y_1, y_2)$ and $\mathcal{G}^{(2)}(x_2 \mid x_1, x_2; y_1, y_2)$ will be given by the sum of Eqs. (4.45)-(4.47). Obviously, in order to make the D-S equation to be closed, the kernel In Eq. (4.45) must be

$$\widetilde{K}(x_1, x_2; z_1, z_2) = K_s(x_1, x_2; z_1, z_2) + K_1(x_1, x_2; z_1, z_2) + K_2(x_1, x_2; z_1, z_2) \qquad (4.48)$$

Thus, the summation of Eqs. (4.45) and (4.47) gives

$$\int d^4 z_1 d^4 z_2 [K_s(x_1, x_2; z_1, z_2) + K_1(x_1, x_2; z_1, z_2) + K_2(x_1, x_2; z_1, z_2)] \mathcal{G}(z_1, z_2; y_1, y_2) \qquad (4.49)$$

Combining Eqs. (4.46) and (4.49), we eventually arrive at

$$\sum_{i=1}^{2} \mathcal{G}^{(i)}(x_i \mid x_1, x_2; y_1, y_2) = \int d^4 z_1 d^4 z_2 K(x_1, x_2; z_1, z_2) \mathcal{G}(z_1, z_2; y_1, y_2) \qquad (4.50)$$

where

$$K(x_1, x_2; z_1, z_2) = \Sigma(x_1, x_2; z_1, z_2) + K_t(x_1, x_2; z_1, z_2)$$
$$+K_s(x_1, x_2; z_1, z_2) + K_1(x_1, x_2; z_1, z_2) + K_2(x_1, x_2; z_1, z_2) \tag{4.51}$$

This just is the kernel appearing in Eq. (2.38). The five terms on the RHS of Eq. (4.51), as respectively shown in Eqs. (4.12), (4.14), (4.17), (4.28) and (4.43), are only represented in terms of the quark, antiquark and gluon propagators and some kinds of three, four and five-line proper vertices and therefore exhibits a more specific structure of the kernel. The equivalence between the both expressions given in the preceding section and this section for the kernel will be illustrated in section 6 for the one-gluon exchange kernels. The exact proof of the equivalence has been done by the technique of irreducible decomposition of the Green's functions. From the proof, we find that the expression described in this section can surely be obtained from the expression given in the preceding section.

5. Pauli-Schrödinger equation

As mentioned in Introduction, the D-S equations formulated in the Dirac spinor space may be reduced to an equivalent equations represented in the Pauli spinor space with the help of Dirac spinors. Let us start from the equation given in Eq. (2.44). The Dirac spinors are defined as [30]

$$U(\overrightarrow{p}) = \sqrt{\frac{\omega + m}{2\omega}} \left(\begin{array}{c} 1 \\ \frac{\overrightarrow{\sigma} \cdot \overrightarrow{p}}{\omega + m} \end{array} \right) \tag{5.1}$$

$$V(\overrightarrow{p}) = \sqrt{\frac{\omega + m}{2\omega}} \left(\begin{array}{c} -\frac{\overrightarrow{\sigma} \cdot \overrightarrow{p}}{\omega + m} \\ 1 \end{array} \right) \tag{5.2}$$

where $\overrightarrow{\sigma}$ are the Pauli matrices, $\omega = (\overrightarrow{p}^2 + m^2)^{1/2}$, $U(\overrightarrow{p})$ and $V(\overrightarrow{p})$ are the positive energy and negative energy spinors respectively. They satisfy the orthonornality relations

$$U^+(\overrightarrow{p})U(\overrightarrow{p}) = V^+(\overrightarrow{p})V(\overrightarrow{p}) = 1$$
$$U^+(\overrightarrow{p})V(\overrightarrow{p}) = V^+(\overrightarrow{p})U(\overrightarrow{p}) = 0 \tag{5.3}$$

and the completeness relation

$$\Lambda^+(\overrightarrow{p}) + \Lambda^-(\overrightarrow{p}) = 1 \tag{5.4}$$

where $\Lambda^+(\overrightarrow{p})$ and $\Lambda^-(\overrightarrow{p})$ are respectively the positive and negative energy state projection operators defined by

$$\Lambda^+(\overrightarrow{p}) = U(\overrightarrow{p})U^+(\overrightarrow{p}), \ \Lambda^-(\overrightarrow{p}) = V(\overrightarrow{p})V^+(\overrightarrow{p}) \tag{5.5}$$

Define

$$W_a(\overrightarrow{p}) = \left\{ \begin{array}{c} U(\overrightarrow{p}), \ if \ a = +, \\ V(\overrightarrow{p}), \ if \ a = -, \end{array} \right\} \tag{5.6}$$

then, the two fermion spinors can be written as

$$W_{ab}(\overrightarrow{P}, \overrightarrow{q}) = W_a(\overrightarrow{p_1})W_b(\overrightarrow{p_2}) \tag{5.7}$$

With this definition, the completeness relation for two fermion spinors can be represented as

$$\sum_{ab} W_{ab}(\overrightarrow{P}, \overrightarrow{q})W_{ab}^+(\overrightarrow{P}, \overrightarrow{q}) = 1 \tag{5.8}$$

Premiltiplying Eq. (2.44) with $W_{ab}^{+}(\overrightarrow{P}, \overrightarrow{q})$, sandwiching Eq. (5.8) between the kernel $K(P, q, k)$ and the amplitude $\chi_{P\varsigma}(k)$ on the RHS of Eq. (2.44) and applying the Dirac equation

$$h(\overrightarrow{p})W_a(\overrightarrow{p}) = a\omega(\overrightarrow{p})W_a(\overrightarrow{p}) \tag{5.9}$$

we obtain

$$\Delta_{ab}(P, \overrightarrow{q})\phi_{ab}(P, q) = \sum_{cd} \int \frac{d^4k}{(2\pi)^4} K_{abcd}(P, q, k)\phi_{cd}(P, k) \tag{5.10}$$

where $a, b, c, d = \pm 1$, $P = (E, \overrightarrow{P})$,

$$\phi_{ab}(P, q) = W_{ab}^{+}(\overrightarrow{P}, \overrightarrow{q})\chi_{P\varsigma}(q) \tag{5.11}$$

$$\Delta_{ab}(P, \overrightarrow{q}) = E - a\omega_1(\overrightarrow{p_1}) - b\omega_2(\overrightarrow{p_2}) \tag{5.12}$$

$$K_{abcd}(P, q, k) = W_{ab}^{+}(\overrightarrow{P}, \overrightarrow{q})K(P, q, k)W_{cd}(\overrightarrow{P}, \overrightarrow{k}) \tag{5.13}$$

Eq. (5.10) is a set of coupled equations satisfied by the amplitudes $\phi_{ab}(P, q)$ each of which is represented in the Pauli spinor space and of dimension four. In the infinite-dimensional space of the momentum q or k, according to $ab = ++$ and $ab \neq ++$, Eq. (5.10) may be, in the matrix form, separately written as

$$\Delta_{++}(p)\phi_{++}(P) = K_{++++}(P)\phi_{++}(P) + \sum_{cd\neq++} K_{++cd}(P)\phi_{cd}(P) \tag{5.14}$$

and

$$\Delta_{ab}(P)\phi_{ab}(P) = K_{ab++}(P)\phi_{++}(P) + \sum_{cd\neq++} K_{abcd}(P)\phi_{cd}(P) \tag{5.15}$$

where $ab \neq ++$ and the terms related to $\phi_{++}(P)$ have been separated out from the others. Furthermore, In the three-dimensional spinor space spanned by $\phi_{ab}(P)$ with $ab \neq ++$, Eqs. (5.14) and (5.15) may be written in the full matrix form

$$\Delta_{+}(P)\psi(P) = K_{+}(P)\psi(P) + \overline{K}^{t}(P)\phi(P) \tag{5.16}$$

and

$$\phi(P) = \overline{G}(P)\psi(P) + G(P)\phi(P) \tag{5.17}$$

where $\psi(P) = \phi_{++}(P)$, $\Delta_{+}(P) = \Delta_{++}(P)$, $K_{+}(P) = K_{++++}(P)$, while, $\phi(P) = \{\phi_{ab}(P)\}$, $\overline{K}^{t}(P) = \{K_{++cd}(P)\}$, $\overline{G}(P) = \{K_{ab++}(P)/\Delta_{ab}(P)\}$ and $G(P) = \{K_{abcd}(P)/\Delta_{ab}(P)\}$ represent the matrices in the three-dimensional spinor space. Solving the equation (5.17), we obtain

$$\phi(P) = \frac{1}{1 - G(P)}\overline{G}(P)\psi(P) \tag{5.18}$$

Substituting the above expression into Eq. (5.16), we finally arrive at

$$\Delta_{+}(P)\psi(P) = V(P)\psi(P) \tag{5.19}$$

where

$$V(P) = K_{+}(P) + \overline{K}^{t}(P)\frac{1}{1 - G(P)}\overline{G}(P) \tag{5.20}$$

which identifies itself with the interaction Hamiltonian. With the definition

$$\frac{1}{1 - G(P)} = \sum_{n=0}^{\infty} G^{(n)}(P), \tag{5.21}$$

Eq. (5.20) can be written as

$$V(P) = \sum_{n=0}^{\infty} V^{(n)}(P) \tag{5.22}$$

where

$$
\begin{aligned}
V^{(0)}(P) &= K_+(P), \\
V^{(1)}(P) &= \overline{K}^t(P)\overline{G}(P), \\
V^{(2)}(P) &= \overline{K}^t(P)G(P)\overline{G}(P), \\
&\cdots\cdots\cdots
\end{aligned}
\tag{5.23}
$$

Written out explicitly, Eq. (5.19) reads

$$[E - \omega_1(\overrightarrow{p_1}) - \omega_2(\overrightarrow{p_2})]\psi(P, q) = \int \frac{d^4k}{(2\pi)^4} V(P, q, k)\psi(P, k) \tag{5.24}$$

The terms in the interaction Hamiltonian in Eq. (5.23) are specified as

$$V^{(0)}(P, q, k) = K_{++++}(P, q, k) \tag{5.25}$$

$$V^{(1)}(P, q, k) = \sum_{ab \neq ++} \int \frac{d^4l}{(2\pi)^4} \frac{K_{++ab}(P, q, l)K_{ab++}(P, l, k)}{E - a\overline{\omega}_1(\overrightarrow{l}) - b\overline{\omega}_2(\overrightarrow{l})} \tag{5.26}$$

$$V^{(2)}(P, q, k) = \sum_{ab \neq ++} \sum_{cd \neq ++} \int \frac{d^4l_1}{(2\pi)^4} \frac{d^4l_2}{(2\pi)^4} \frac{K_{++ab}(P, q, l_1)K_{abcd}(P, l_1, l_2)K_{cd++}(P, l_2, k)}{[E - a\overline{\omega}_1(\overrightarrow{l_1}) - b\overline{\omega}_2(\overrightarrow{l_1})][E - a\overline{\omega}_1(\overrightarrow{l_2}) - b\overline{\omega}_2(\overrightarrow{l_2})]} \tag{5.27}$$

and so on, where for simplicity of representation, we have defined $\overline{\omega}_1(\overrightarrow{l}) = \omega_1(\eta_1\overrightarrow{P} + \overrightarrow{l})$ and $\overline{\omega}_2(\overrightarrow{l}) = \omega_2(\eta_2\overrightarrow{P} - \overrightarrow{l})$. In the center of mass frame, $\overline{\omega}_i(\overrightarrow{l}) = \omega_i(\overrightarrow{l})$ $(i = 1, 2)$. Eq. (5.24) is the equation satisfied by the positive energy state amplitude $\psi(P, q)$ which is of dimension four in the two-fermion Pauli spinor space. This is the reason why the above equation is called Pauli-Schrödinger (P-S) equation.

By the same procedure, the D-S equation in Eq. (2.45) may also be reduced to a corresponding P-S equation as represented in the following

$$[q_0 - \eta_2\omega_1(\overrightarrow{p_1}) + \eta_1\omega_2(\overrightarrow{p_2})]\psi(P, q) = \int \frac{d^4k}{(2\pi)^4} \overline{V}(P, q, k)\psi(P, k) \tag{5.28}$$

where q_0 is the relative energy and $\overline{V}(P, q, k)$ is a kind of interaction Hamiltonian which can be written out from the expression of $V(P, q, k)$ by the replacement: $K(P, q, k) \to \overline{K}(P, q, k)$ and $\Delta_{ab}(P, \overrightarrow{q}) \to \overline{\Delta}_{ab}(\overrightarrow{P}, q) = q_0 - a\eta_2\omega_1(\overrightarrow{p_1}) + b\eta_1\omega_2(\overrightarrow{p_2})$. For the three-dimensional D-S equation, the P-S equation in Eq. (5.28) disappears. We are left only with a three-dimensional P-S equation derived from Eq. (2.56) such that

$$[E - \omega_1(\overrightarrow{p_1}) - \omega_2(\overrightarrow{p_2})]\psi(P, \overrightarrow{q}) = \int \frac{d^3k}{(2\pi)^3} V(P, \overrightarrow{q}, \overrightarrow{k})\psi(P, \overrightarrow{k}) \tag{5.29}$$

where the Hamiltonian $V(P, \overrightarrow{q}, \overrightarrow{k})$ formally has the same expressions as written in Eqs. (5.22) and (5.25)-(5.27) except that the four-dimensional kernel $K(P, q, k)$ in those expressions is now replaced by the three-dimensional one $K(P, \overrightarrow{q}, \overrightarrow{k})$ which is the Fourier transform of the kernel in Eq. (3.40)

It is worthy to point out that for a given kernel in the D-S equation, there are a series of terms (the ladder diagrams) to appear in the interaction Hamiltonian in the P-S equation. If the D-S equation with a given kernel could be solved, the contribution arising from a series of ladder diagrams characterized by the series of terms in the Hamiltonian are precisely taken into account. Another point we would like to stress is that as seen from Eqs. (5.26) and (5.27), the negative energy state only acts as intermediate states to appear in the interaction Hamiltonian. Particularly, for the bound state, the positive energy state does not appear in the intermediate states. While, for the scattering state P-S equation as discussed in Ref. [30], the intermediate states in the interaction Hamiltonian must include the positive energy state. In this case, the series expansion of the interaction Hamiltonian in Eq. (5.22) has an one-to-one correspondence with the perturbative expansion of the S-matrix. The above statement reveals an essential difference between the interactions taking place in the bound state and the scattering state.

6. One-gluon exchange kernels

In this section, we limit ourself to give a brief derivation and description of the one gluon exchange kernel (OGEK). First we discuss the t-channel OGEK and the s-channel OGEK appearing in the four-dimensional D-S equation to illustrate the equivalence between the expressions of the interaction kernel derived in the sections 3 and 4. Then, we show the OGEK in the three-dimensional D-S equation and the corresponding Hamiltonian in the P-S equation.

A. The t-channel one-gluon exchange kernel

The exact form of the t-channel OGEK was represented in Eq. (4.14) with $\Sigma_\mu^a(x_2 \mid x_1; z_1)$ and $\Sigma_\mu^{ca}(x_1 \mid x_2; z_2)$ given in Eqs. (4.6) and (4.9). In the lowest-order approximation, the propagators and the vertices in Eqs. (4.6) and (4.9) are taken respectively to be the free ones and the bare ones. The bare vertices are of the form

$$\begin{aligned}
&\Gamma^{b\nu}(u_1 \mid u_2, z_1) = -ig\gamma^\nu T^b \delta^4(u_1 - u_2)\delta^4(u_2 - z_1) \\
&\Gamma_c^{b\nu}(u_1 \mid u_2, z_2) = -ig\gamma^\nu \overline{T}^b \delta^4(u_1 - u_2)\delta^4(u_2 - z_2)
\end{aligned} \tag{6.1}$$

With the vertices given above, the kernel in Eq. (4.14) becomes

$$\begin{aligned}
K_t^0(X - Y, x, y) = ig^2 T^a \overline{T}^b \{ &\Delta_{\mu\nu}^{ab}(x_2 - y_1) S_F(x_1 - y_1)\gamma_1^\mu \gamma_2^0 \gamma_2^\nu \delta^4(x_2 - y_2) \\
+ &\Delta_{\mu\nu}^{ab}(x_1 - y_2) S_F^c(x_2 - y_2)\gamma_2^\nu \gamma_1^0 \gamma_1^\mu \delta^4(x_1 - y_1) \}
\end{aligned} \tag{6.2}$$

From now on, the $S_F(x - y)$ and $\Delta_{\mu\nu}^{ab}(x - y)$ in the above are understood to be free propagators. By the Fourier transformation, we get in the momentum space that

$$K_t^0(P, q, k) = S(P, q)ig^2 T^a \overline{T}^b \Delta_{\mu\nu}^{ab}(q - k)\gamma_1^\mu \gamma_2^\nu \tag{6.3}$$

where

$$S(P, q) = S_F(p_1)\gamma_2^0 + S_F^c(p_2)\gamma_1^0 = [\widehat{S}_F(p_1) + \widehat{S}_F^c(p_2)]\gamma_1^0 \gamma_2^0 \tag{6.4}$$

in which

$$S_F(p) = \widehat{S}_F(p)\gamma^0 \tag{6.5}$$

$$\widehat{S}_F(p) = \frac{1}{p_0 - h(\overrightarrow{p}) + i\varepsilon} = \frac{\Lambda^+(\overrightarrow{p})}{p_0 - \omega(\overrightarrow{p}) + i\varepsilon} + \frac{\Lambda^-(\overrightarrow{p})}{p_0 + \omega(\overrightarrow{p}) - i\varepsilon} \tag{6.6}$$

here $h(\overrightarrow{p})$ is the free fermion Hamiltonian, $\Lambda^+(\overrightarrow{p})$ and $\Lambda^-(\overrightarrow{p})$ were defined in Eq. (5.5).

Let us turn to derive the above kernel from the closed expression presented in Eq. (3.30). In the perturbative approximation of order g^2, only the first and second terms in Eq. (3.30) can contribute to the OGEK. In the first

term which was defined in Eqs. (3.31), (3.23) and (3.21), there are four terms: two represent the self-energies of quark and antiquark and the other two are related to the t-channel one-gluon exchange interaction which is concerned here only. The $\Lambda_\mu^a(x_2 \mid x_1, z_1)$ and $\Lambda_\mu^{ca}(x_1 \mid x_2, z_2)$ in Eq. (3.21) was respectively represented in Eqs. (4.5), (4.6), (4.8) and (4.9). When the vertices are taken to be the bare ones shown in Eq. (6.1), the terms contained in the $\mathcal{R}(x_1, x_2; z_1, z_2)$ which contributes to the OGEK can be written as

$$
\begin{aligned}
H_1^t(x_1, x_2; z_1, z_2) = ig^2 T^a \overline{T}^b \int d^4 u \{ \Delta_{\mu\nu}^{ab}(x_2 - u)\gamma_2^0 \gamma_2^\mu \gamma_2^0 S_F(x_1 - u)\gamma_1^\nu S_F(u - z_1)\delta^4(x_2 - z_2) \\
+ \Delta_{\mu\nu}^{ab}(x_1 - u)\gamma_1^0 \gamma_1^\mu \gamma_1^0 S_F^c(x_2 - u)\gamma_2^\nu S_F^c(u - z_2)\delta^4(x_1 - z_1) \}
\end{aligned}
\tag{6.7}
$$

Next, we turn to the second term in Eq. (3.30) which was defined in Eqs. (3.32) and (3.14). Through a perturbative calculation of the Green's function $\mathcal{G}_{\mu\nu}^{ab}(x_i, z_j \mid x_1, x_2; z_1, z_2)$ defined in Eq. (3.14) or performing a decomposition of the Green's function into the connected ones, one may find that there are a function $i\Delta_{\mu\nu}^{ab}(x_i - z_j)S_F(x_1 - z_1)S_F^c(x_2 - z_2)$ included in the function $\mathcal{G}_{\mu\nu}^{ab}(x_i, z_j \mid x_1, x_2; z_1, z_2)$ which just is related to the t-channel OGEK when $i \neq j$. Thus, according to Eq. (3.32), the term included in the $\mathcal{Q}(x_1, x_2; z_1, z_2)$ which contributes to the OGEK may be written as:

$$
\begin{aligned}
H_2^t(x_1, x_2; z_1, z_2) = ig^2 T^a \overline{T}^b \{ \Delta_{\mu\nu}^{ab}(x_1 - z_2)\gamma_1^0 \gamma_1^\mu S_F(x_1 - z_1)S_F^c(x_2 - z_2)\gamma_2^\nu \gamma_2^0 \\
+ \Delta_{\mu\nu}^{ab}(x_2 - z_1)\gamma_2^0 \gamma_2^\mu S_F^c(x_2 - z_2)S_F(x_1 - z_1)\gamma_1^\nu \gamma_1^0 \}
\end{aligned}
\tag{6.8}
$$

Substituting Eqs. (6.7) and (6.8) into Eq. (3.30), we will obtain the expression of the t-channel OGEK $K_t^0(X-Y, x, y)$. By Fourier transformation, its expression given in the momentum space may be found to be

$$
K_t^0(P, q, k) = \sum_{i=1}^2 H_i^t(P, q, k)S^{-1}(P, k)
\tag{6.9}
$$

where

$$
\begin{aligned}
H_1^t(P, q, k) = ig^2 T^a \overline{T}^b \Delta_{\mu\nu}^{ab}(q - k)\{ \gamma_2^0 \gamma_2^\mu \gamma_2^0 S_F(p_1)\gamma_1^\nu S_F(q_1) \\
+ \gamma_1^0 \gamma_1^\mu \gamma_1^0 S_F^c(p_2)\gamma_2^\nu S_F^c(q_2) \}
\end{aligned}
\tag{6.10}
$$

and

$$
\begin{aligned}
H_2^t(P, q, k) = ig^2 T^a \overline{T}^b \Delta_{\mu\nu}^{ab}(q - k)\{ \gamma_1^0 \gamma_1^\mu S_F(q_1)S_F^c(p_2)\gamma_2^\nu \gamma_2^0 \\
+ \gamma_2^0 \gamma_2^\mu S_F^c(q_2)S_F(p_1)\gamma_1^\nu \gamma_1^0 \}
\end{aligned}
\tag{6.11}
$$

Employing the representation of fermion propagator denoted in Eqs. (6.5) and (6.6) and noticing

$$
S^{-1}(P, k) = \gamma_1^0 \gamma_2^0 [\widehat{S}_F(q_1) + \widehat{S}_F^c(q_2)]^{-1}
\tag{6.12}
$$

one may exactly obtain from Eqs. (6.9)-(6.11) the expression denoted in Eq. (6.3). Thus, the equivalence between the both expressions of the D-S kernel derived in sections 3 and 4 is proved in the lowest order approximation.

Now let us focus on the three-dimensional t-channel OGEK which was derived for the first time in Ref. [15, 16]. For comparison with the four-dimensional kernel, it is necessary to give this kernel a further description based on the closed expression formulated in Eqs. (3.40)-(3.47). Analogous to the four-dimensional case, in the lowest order approximation, only the first two terms in Eq. (3.40) can contribute to the three-dimensional t-channel OGEK. Therefore, we can write

$$
K_t^0(\vec{x_1}, \vec{x_2}; \vec{y_1}, \vec{y_2}; t_1 - t_2) = \int d^3 z_1 d^3 z_2 \sum_{i=1}^2 H_i^t(\vec{x_1}, \vec{x_2}; \vec{z_1}, \vec{z_2}; t_1 - t_2)S^{-1}(\vec{z_1}, \vec{z_2}; \vec{y_1}, \vec{y_2})
\tag{6.13}
$$

where $H_1^t(\vec{x_1}, \vec{x_2}; \vec{z_1}, \vec{z_2}; t_1 - t_2)$ arises from Eqs. (3.41) and (3.42) with the three-point Green's functions in Eq. (3.42) being given by

$$
\begin{aligned}
\Lambda_\mu^a(\vec{x_2} \mid \vec{x_1}, \vec{z_1}; t_1 - t_2) = -ig \int d^3 u du_0 \Delta_{\mu\nu}^{ab}(\vec{x_2} - \vec{u}; t_1 - u_0) \\
\times S_F(\vec{x_1} - \vec{u}; t_1 - u_0)\gamma^\nu T^b S_F(\vec{u} - \vec{z_1}; u_0 - t_2) \}
\end{aligned}
\tag{6.14}
$$

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$$\Lambda_\mu^{\mathbf{c}a}(\overrightarrow{x_1} \mid \overrightarrow{x_2}, \overrightarrow{z_2}; t_1 - t_2) = -ig \int d^3 u du_0 \Delta_{\mu\nu}^{ab}(\overrightarrow{x_1} - \overrightarrow{u}; t_1 - u_0)$$
$$\times S_F^c(\overrightarrow{x_2} - \overrightarrow{u}; t_1 - u_0)\gamma^\nu \overline{T}^b S_F^c(\overrightarrow{u} - \overrightarrow{z_2}; u_0 - t_2)\}$$
(6.15)

and $H_2^t(\overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}; t_1 - t_2)$ is derived from Eqs. (3.43) and (3.44) when the terms $i\Delta_{\mu\nu}^{ab}(\overrightarrow{x_1} - \overrightarrow{z_1}; t_1 - t_2)S_F(\overrightarrow{x_1} - \overrightarrow{z_1}; t_1 - t_2)S_F^c(\overrightarrow{x_2} - \overrightarrow{z_2}; t_1 - t_2)$ included in $\mathcal{G}_{\mu\nu}^{ab}(\overrightarrow{x_i}, \overrightarrow{z_j} \mid \overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}; E)$ with $i \neq j$ are taken into account only. By the Fourier transformation, it is not difficult to derive the following expression

$$K_t^0(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k}, E) = \sum_{i=1}^{2} H_i^t(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k}, E)S^{-1}(\overrightarrow{P}, \overrightarrow{k})$$
(6.16)

where

$$H_1^t(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k}, E)$$
$$= -ig \int \frac{dq_0}{2\pi} \frac{dk_0}{2\pi} \{\Omega_1^{a\mu} \gamma_1^0 \Delta_{\mu\nu}^{ab}(\overrightarrow{q} - \overrightarrow{k}; q_0 - k_0)S_F^c(\overrightarrow{p_2}, q_0)\gamma^\nu \overline{T}^b S_F^c(\overrightarrow{q_2}, k_0)$$
$$+ \Omega_2^{a\mu} \gamma_2^0 \Delta_{\mu\nu}^{ab}(\overrightarrow{k} - \overrightarrow{q}; k_0 - q_0)S_F(\overrightarrow{p_1}, q_0)\gamma^\nu T^b S_F(\overrightarrow{q_1}, k_0)\}$$
(6.17)

and

$$H_2^t(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k}, E)$$
$$= i \int \frac{dq_0}{2\pi} \frac{dk_0}{2\pi} \{\Omega_1^{a\mu} \Delta_{\mu\nu}^{ab}(\overrightarrow{q} - \overrightarrow{k}; E - q_0 - k_0)S_F(\overrightarrow{q_1}, k_0)S_F^c(\overrightarrow{p_2}, q_0)\overline{\Omega}_2^{b\nu}$$
$$+ \Omega_2^{a\mu} \Delta_{\mu\nu}^{ab}(\overrightarrow{k} - \overrightarrow{q}; E - k_0 - q_0)S_F(\overrightarrow{p_1}, q_0)S_F^c(\overrightarrow{q_2}, k_0)\overline{\Omega}_1^{b\nu}\}$$
(6.18)

The integrals over q_0 and k_0 can easily be calculated by applying the Cauchy theorem in the complex planes of q_0 and k_0. Since QCD is an unitary theory, the matrix element of the kernel between the spinor wave functions is independent of the gauge parameter. Therefore, we only need to show the result given in the Feynman gauge. In this gauge, noticing the representation of the gluon propagator

$$\Delta_{\mu\nu}^{ab}(Q) = -\frac{\delta_{ab}g_{\mu\nu}}{Q_0^2 - \overrightarrow{Q}^2 + i\varepsilon}$$
$$= -\frac{\delta_{ab}g_{\mu\nu}}{2\left|\overrightarrow{Q}\right|}\left[\frac{1}{Q_0 - \left|\overrightarrow{Q}\right| + i\varepsilon} - \frac{1}{Q_0 + \left|\overrightarrow{Q}\right| - i\varepsilon}\right]$$
(6.19)

and the expression shown in Eqs. (6.5) and (6.6), it can be found that

$$H_1^t(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k}, E) = \frac{ig^2 T_1^a \overline{T}_2^a}{2\left|\overrightarrow{q} - \overrightarrow{k}\right|}\{\frac{1}{\omega(\overrightarrow{p_2}) + \omega(\overrightarrow{q_2}) + \left|\overrightarrow{q} - \overrightarrow{k}\right|}\gamma_1^0\gamma_1^\mu[\Lambda^+(\overrightarrow{p_2})\gamma_2^0\gamma_{2\mu}\Lambda^-(\overrightarrow{q_2})$$
$$+ \Lambda^-(\overrightarrow{p_2})\gamma_2^0\gamma_{2\mu}\Lambda^+(\overrightarrow{q_2})] + \frac{1}{\omega(\overrightarrow{p_1}) + \omega(\overrightarrow{q_1}) + \left|\overrightarrow{q} - \overrightarrow{k}\right|}$$
$$\times \gamma_2^0\gamma_2^\mu[\Lambda^+(\overrightarrow{p_1})\gamma_1^0\gamma_{1\mu}\Lambda^-(\overrightarrow{q_1}) + \Lambda^-(\overrightarrow{p_1})\gamma_1^0\gamma_{1\mu}\Lambda^+(\overrightarrow{q_1})]\}\gamma_1^0\gamma_2^0$$
(6.20)

and

$$H_2^t(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k}, E) = \frac{ig^2 T_1^a \overline{T}_2^a}{2\left|\overrightarrow{q} - \overrightarrow{k}\right|}\{\frac{\Lambda^+(\overrightarrow{p_2})\gamma_2^0\gamma_{2\mu}\gamma_1^0\gamma_1^\mu\Lambda^+(\overrightarrow{q_1})}{E - \omega(\overrightarrow{p_2}) - \omega(\overrightarrow{q_1}) - \left|\overrightarrow{q} - \overrightarrow{k}\right|}$$
$$+ \frac{\Lambda^+(\overrightarrow{p_1})\gamma_1^0\gamma_1^\mu\gamma_2^0\gamma_{2\mu}\Lambda^+(\overrightarrow{q_2})}{E - \omega(\overrightarrow{p_1}) - \omega(\overrightarrow{q_2}) - \left|\overrightarrow{q} - \overrightarrow{k}\right|} - \frac{\Lambda^-(\overrightarrow{p_2})\gamma_2^0\gamma_{2\mu}\gamma_1^0\gamma_1^\mu\Lambda^-(\overrightarrow{q_1})}{E + \omega(\overrightarrow{p_2}) + \omega(\overrightarrow{q_1}) + \left|\overrightarrow{q} - \overrightarrow{k}\right|}$$
$$- \frac{\Lambda^-(\overrightarrow{p_1})\gamma_1^0\gamma_1^\mu\gamma_2^0\gamma_{2\mu}\Lambda^-(\overrightarrow{q_2})}{E + \omega(\overrightarrow{p_1}) + \omega(\overrightarrow{q_2}) + \left|\overrightarrow{q} - \overrightarrow{k}\right|}\}\gamma_1^0\gamma_2^0$$
(6.21)

It is seen that $H_1^t(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k}, E)$ is actually independent of the energy E. We would like to emphasize that the expressions in Eqs. (6.20) and (6.21) can more directly be obtained from Eqs. (6.10) and (6.11) by the following integration

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$$H_i^t(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k}, E) = \int \frac{dq_0}{2\pi} \frac{dk_0}{2\pi} H_i^t(P, q, k), i = 1, 2. \tag{6.22}$$

Now, we discuss the inverse of the function $S(\overrightarrow{P}, \overrightarrow{k})$ which is the Fourier transform of the function in Eq. (2.50). The equal-time propagators can be defined in such a manner [34]

$$S_F(\overline{x} - \overline{y}) = \frac{1}{2i} \left\langle 0^+ \left| \psi(\overline{x}, t)\overline{\psi}(\overline{y}, t) - \overline{\psi}^T(\overline{y}, t)\psi^T(\overline{x}, t) \right| 0^- \right\rangle$$
$$= \int \frac{d^3p}{(2\pi)^3} S_F(\overline{p}) e^{i\overrightarrow{p}\cdot(\overline{x} - \overline{y})} \tag{6.23}$$

where

$$S_F(\overline{p}) = \frac{1}{2i} \frac{h(\overline{p})}{\omega(\overline{p})} \gamma^0 \tag{6.24}$$

With this representation, the function $S(\overrightarrow{P}, \overrightarrow{k})$ and its inverse, i.e. the three-dimensional counterparts of those in Eqs. (6.4) and (6.12) will be written as

$$S(\overrightarrow{P}, \overrightarrow{k}) = \frac{1}{2i}[\frac{h(\overrightarrow{q_1})}{\omega(\overrightarrow{q_1})} + \frac{h(\overrightarrow{q_2})}{\omega(\overrightarrow{q_2})}]\gamma_1^0\gamma_2^0 \tag{6.25}$$

and

$$S^{-1}(\overrightarrow{P}, \overrightarrow{k}) = 2i\gamma_1^0\gamma_2^0[\frac{h(\overrightarrow{q_1})}{\omega(\overrightarrow{q_1})} + \frac{h(\overrightarrow{q_2})}{\omega(\overrightarrow{q_2})}]^{-1} \tag{6.26}$$

When Eqs. (6.20), (6.21) and (6.26) are substituted into Eq. (6.16), one may write out explicitly the expression of the three-dimensional t-channel OGEK. On inserting this kernel into the first term of the effective interaction Hamiltonian denoted in Eq. (5.25) and employing the orthogonality relations of Dirac spinors and the Dirac equation, we are led to [15, 16]

$$V_t^{(0)}(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k}; \overrightarrow{E}) = g^2 T_1^a \overline{T}_2^a \Delta(\overrightarrow{q} - \overrightarrow{k}; E)\overline{U}(\overrightarrow{p_1})\gamma_1^\mu U(\overrightarrow{q_1})\overline{U}(\overrightarrow{p_2})\gamma_{2\mu}U(\overrightarrow{q_2}) \tag{6.27}$$

where $U(\overrightarrow{q})$ was represented in Eq. (5.1) and

$$\Delta(\overrightarrow{q} - \overrightarrow{k}; E) = -\frac{1}{2\left|\overrightarrow{q} - \overrightarrow{k}\right|}[\frac{1}{E - \omega(\overrightarrow{p_2}) - \omega(\overrightarrow{q_1}) - \left|\overrightarrow{q} - \overrightarrow{k}\right|}$$
$$+ \frac{1}{E - \omega(\overrightarrow{p_1}) - \omega(\overrightarrow{q_2}) - \left|\overrightarrow{q} - \overrightarrow{k}\right|}] \tag{6.28}$$

is just the exact three-dimensional gluon propagator given in the Feynman gauge which is off-shell because the energy E is off-shell. It is noted here that the lowest order interaction Hamiltonian $V^{(0)}(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k}; \overrightarrow{E})$ is only given by the function $H_2^t(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k}, E)$ in Eq. (6.21) because the function $H_1^t(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k}, E)$ in Eq. (6.20) gives a vanishing contribution to the lowest order Hamiltonian.

B. The s-channel one-gluon exchange kernel

The four-dimensional s-channel OGEK was represented in Eq. (4.17). By means of the charge conjugation of the quark field, the vertex in Eq. (4.17) can be expressed as

$$\overline{\Gamma}^{b\nu*}(z \mid z_1, z_2)_{\rho\sigma} = -C_{\sigma\lambda}\Gamma^{b\nu}(z \mid z_2, z_1)_{\lambda\rho} \tag{6.29}$$

With this relation and the expression shown in Eq. (6.1), the kernel in the lowest-order approximation can be written as

$$K_s^0(x_1, x_2; z_1, z_2)_{\alpha\beta\rho\sigma} = ig\{\Delta_{\mu\nu}^{ab}(x_1 - z_1)(\Omega_1^{a\mu})_{\alpha\gamma} S_F^*(x_1 - x_2)_{\gamma\beta}$$
$$+\Delta_{\mu\nu}^{ab}(x_2 - z_2)(\Omega_2^{a\mu})_{\beta\lambda} S_F^*(x_1 - x_2)_{\alpha\lambda}\}(C\gamma^\nu T^b)_{\rho\sigma}\delta^4(z_1 - z_2) \tag{6.30}$$

In the momentum space, it reads

$$K_s^0(P, q, k)_{\alpha\beta\rho\sigma} = ig\Delta_{\mu\nu}^{ab}(P)L^{a\mu}(P, q)_{\alpha\beta}(C\gamma^\nu T^b)_{\rho\sigma} \tag{6.31}$$

where

$$L^{a\mu}(P, q)_{\alpha\beta} = (\Omega_1^{a\mu})_{\alpha\gamma} S_F^*(-p_2)_{\gamma\beta} + (\Omega_2^{a\mu})_{\beta\lambda} S_F^*(p_1)_{\alpha\lambda} \tag{6.32}$$

in which

$$S_F^*(p) = \int d^4x S_F^*(x)e^{-iqx} = S_F(p)C^{-1} = C^{-1}S_F^c(-p)^T \tag{6.33}$$

Now, let us derive the above kernel from the closed expression in Eq. (3.30). From the perturbative calculation, It can be found that in the lowest order approximation, only the second term in Eq. (3.30) can contribute to the s-channel OGEK because in the perturbative expansion of the Green's function $\mathcal{G}_{\mu\nu}^{ab}(x_i, z_j \mid x_1, x_2; z_1, z_2)$, there is a term $-i\Delta_{\mu\nu}^{ab}(x_i - z_j)S_F^*(x_1 - x_2)\overline{S}_F^*(z_1 - z_2)$ which is merely related to the s-channel OGEK. Thus, the terms in the $\mathcal{Q}(x_1, x_2; z_1, z_2)$ which contribute to the s-channel OGEK, according to Eq. (3.32) can be written as

$$H^s(x_1, x_2; z_1, z_2) = -\sum_{i,j=1}^{2} i\Delta_{\mu\nu}^{ab}(x_i - z_j)\Omega_i^{a\mu} S_F^*(x_1 - x_2)\overline{S}_F^*(z_1 - z_2)\overline{\Omega}_j^{b\nu} \tag{6.34}$$

Substituting the above expression into Eq. (3.30), in the momentum space, we have

$$K_s^0(P, q, k) = H^s(P, q, k)S^{-1}(P, k) \tag{6.35}$$

where

$$H^s(P, q, k)_{\alpha\beta\lambda\delta} = i\Delta_{\mu\nu}^{ab}(P)L^{a\mu}(P, q)_{\alpha\beta}\overline{L}^{b\nu}(P, k)_{\lambda\delta} \tag{6.36}$$

in which $L^{a\mu}(P, q)_{\alpha\beta}$ was given in (6.32) and

$$\overline{L}^{b\nu}(P, k)_{\lambda\delta} = -[\overline{S}_F^*(-q_1)_{\lambda\tau}(\overline{\Omega}_2^{a\mu})_{\tau\delta} + \overline{S}_F^*(q_2)_{\tau\delta}(\overline{\Omega}_1^{a\mu})_{\tau\lambda}] \tag{6.37}$$

here

$$\overline{S}_F^*(q) = \int d^4x \overline{S}_F^*(x)e^{iqx} = C^{-1}S_F^c(q) = S_F^T(-q)C^{-1} \tag{6.38}$$

In light of the charge conjugation for the γ-matrix and for the propagators shown in Eqs. (6.33) and (6.38), it is easy to find

$$\overline{L}^{b\nu}(P, k)_{\lambda\delta}S^{-1}(P, k)_{\lambda\delta\rho\sigma} = g(C\gamma^\nu T^b)_{\rho\sigma} \tag{6.39}$$

With this relation, we see, the kernel in Eq. (6.35) is exactly equal to the one written in Eq. (6.31). This gives a further proof of the equivalence between the both expressions of the D-S kernel derived in sections 3 and 4. By the charge conjugation, it is not difficult to find

$$L^{a\mu}(P, q)_{\alpha\beta} = g\widehat{S}(P, q)_{\alpha\beta\lambda\tau}(C\gamma^\mu T^a)_{\lambda\tau} \tag{6.40}$$

where

$$\widehat{S}(P, q) = \widehat{S}_F(p_1) + \widehat{S}_F^c(p_2) \tag{6.41}$$

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Therefore, the kernel in Eq. (6.31) can be expressed as

$$K_s^0(P, q, k) = ig^2 \Delta_{\mu\nu}^{ab}(P) \widehat{S}(P, q)_{\alpha\beta\lambda\tau} (C\gamma^\mu T^a)_{\lambda\tau} (C\gamma^\nu T^b)_{\rho\sigma} \tag{6.42}$$

At the last of this section, we would like to discuss the three-dimensional form of the s-channel OGEK. In accordance with Eq. (3.40), this kernel is represented as

$$K_s^0(\overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{y_1}, \overrightarrow{y_2}; t_1 - t_2) = \int d^3 z_1 d^3 z_2 H^s(\overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}; t_1 - t_2) S^{-1}(\overrightarrow{z_1}, \overrightarrow{z_2}; \overrightarrow{y_1}, \overrightarrow{y_2}) \tag{6.43}$$

where $H^s(\overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}; t_1 - t_2)$ can be written from Eq. (6.34) by setting $x_1^0 = x_2^0 = t_1$ and $z_1^0 = z_2^0 = t_2$ in the equal-time frame, that is

$$H^s(\overrightarrow{x_1}, \overrightarrow{x_2}; \overrightarrow{z_1}, \overrightarrow{z_2}; t_1 - t_2) = -i \sum_{i,j=1}^2 \Delta_{\mu\nu}^{ab}(\overrightarrow{x}_i - \overrightarrow{z}_j; t_1 - t_2) \Omega_i^{a\mu} S_F^*(\overrightarrow{x}_1 - \overrightarrow{x}_2) \overline{S}_F^*(\overrightarrow{z}_1 - \overrightarrow{z}_2) \overline{\Omega}_j^{b\nu} \tag{6.44}$$

In the momentum space, it is of the form

$$H^s(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k}; E)_{\alpha\beta\lambda\delta} = i\Delta_{\mu\nu}^{ab}(\overrightarrow{P}, E) L^{a\mu}(\overrightarrow{P}, \overrightarrow{q})_{\alpha\beta} \overline{L}^{b\nu}(\overrightarrow{P}, \overrightarrow{k})_{\lambda\delta} \tag{6.45}$$

where

$$L^{a\mu}(\overrightarrow{P}, \overrightarrow{q})_{\alpha\beta} = (\Omega_1^{a\mu})_{\alpha\gamma} S_F^*(-\overrightarrow{p}_2)_{\gamma\beta} + (\Omega_2^{a\mu})_{\beta\gamma} S_F^*(\overrightarrow{p}_1)_{\alpha\gamma} \tag{6.46}$$

and

$$\overline{L}^{b\nu}(\overrightarrow{P}, \overrightarrow{k})_{\lambda\delta} = -[\overline{S}_F^*(-\overrightarrow{q}_1)_{\lambda\tau} (\overline{\Omega}_2^{a\mu})_{\tau\delta} + \overline{S}_F^*(\overrightarrow{q}_2)_{\tau\delta} (\overline{\Omega}_1^{a\mu})_{\tau\lambda}] \tag{6.47}$$

which are the three-dimensional form of the functions in Eqs. (6.32) and (6.37). It is emphasized that in Eq. (6.45), only the gluon propagator is dependent on energy E, while, the fermion propagators are energy-independent. By the same charge conjugation transformations as shown in Eqs. (6.33) and (6.38), one may obtain a kernel similar to Eq. (6.42)

$$K_s^0(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k}; E)_{\alpha\beta\rho\sigma} = ig^2 \Delta_{\mu\nu}^{ab}(\overrightarrow{P}, E) \widehat{S}(\overrightarrow{P}, \overrightarrow{q})_{\alpha\beta\lambda\tau} (C\gamma^\mu T^a)_{\lambda\tau} (C\gamma^\nu T^b)_{\rho\sigma} \tag{6.48}$$

which may also be represented as

$$K_s^0(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k}; E)_{\alpha\beta\rho\sigma} = ig^2 \Delta_{\mu\nu}^{ab}(\overrightarrow{P}, E) S(\overrightarrow{P}, \overrightarrow{q})_{\alpha\beta\lambda\tau} (\gamma^\mu C T^a)_{\lambda\tau} (C\gamma^\nu T^b)_{\rho\sigma} \tag{6.49}$$

where

$$S(\overrightarrow{P}, \overrightarrow{q}) = \widehat{S}(\overrightarrow{P}, \overrightarrow{q}) \gamma_1^0 \gamma_2^0 \tag{6.50}$$

which is the three-dimensional form of Eq. (6.4).

In the P-S equation, similar to Eq. (6.27), the lowest order interaction Hamiltonian given by the kernel in Eq. (6.48), according to Eq. (5.25), will be written as

$$V_s^{(0)}(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k}; E) = g^2 \Delta_{\mu\nu}^{ab}(\overrightarrow{P}, E) U_\alpha(\overrightarrow{p_1})^+ U_\beta(\overrightarrow{p_2})^+ (C\gamma^\mu T^a)_{\alpha\beta} (C\gamma^\nu T^b)_{\rho\sigma} U_\rho(\overrightarrow{q_1}) U_\sigma(\overrightarrow{q_2}) \tag{6.51}$$

It should be noted that the positive energy state Dirac spinors used here were defined in Eq. (5.1). The negative energy state spinor may be given by the charge conjugation relation: $V(\overrightarrow{p}) = CU(\overrightarrow{p})$ here $C = \gamma^5\gamma^0$ [30]. However, the matrix C in Eq. (6.51) is defined by $C = i\gamma^2\gamma^0$ [18]. Correspondingly, the charge conjugation relation between the spinor wave functions is given by $v^s(\overrightarrow{p}) = C\overline{u}^s(\overrightarrow{p})^T$ where $\overline{u}^s(\overrightarrow{p}) = u^s(\overrightarrow{p})^+\gamma^0$ with $u^s(\overrightarrow{p})$ and $v^s(\overrightarrow{p})$ being the positive and negative energy spinor wave functions respectively [18] and represented as

$$u^s(\overrightarrow{p}) = \widetilde{U}(\overrightarrow{p}) \varphi^s(\overrightarrow{p}), v^s(\overrightarrow{p}) = \widetilde{V}(\overrightarrow{p}) \chi^s(\overrightarrow{p}) \tag{6.52}$$

here $\varphi^s(\overrightarrow{p})$ and $\chi^s(\overrightarrow{p})$ are the spin wave functions and

$$\widetilde{U}(\overrightarrow{p}) = \sqrt{\frac{\omega}{m}} U(\overrightarrow{p}), \widetilde{V}(\overrightarrow{p}) = -\sqrt{\frac{\omega}{m}} \gamma^0 V(\overrightarrow{p}) \tag{6.53}$$

Usually, the S-matrix element given by the kernel in Eq. (6.48) is represented by

$$T_s(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k; E}) = u_\alpha^{s_1}(\overrightarrow{p_1})^+ u_\beta^{s_2}(\overrightarrow{p_2})^+ K_s^0(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k}; E)_{\alpha\beta\rho\sigma} u_\rho^{r_1}(\overrightarrow{q_1}) u_\sigma^{r_2}(\overrightarrow{q_2}) \tag{6.54}$$

For later derivation, it is more convenient to use the expression of the kernel written in Eq. (6.49). On inserting Eq. (6.49) into Eq. (6.54) and noticing

$$u_\alpha^{s_1}(\overrightarrow{p_1})^+ u_\beta^{s_2}(\overrightarrow{p_2})^+ S(\overrightarrow{P}, \overrightarrow{q})_{\alpha\beta\lambda\tau} = -i\overline{u}_\lambda^{s_1}(\overrightarrow{p_1}) \overline{u}_\tau^{s_2}(\overrightarrow{p_2}) \tag{6.55}$$

which is obtained by applying the Dirac equation and

$$\begin{aligned} &\overline{u}_\beta^{s_2}(\overrightarrow{p_2})(C\gamma^\mu T^a)_{\alpha\beta} = (\gamma^\mu T^a)_{\alpha\beta} v_\beta^{s_2}(\overrightarrow{p_2}), \\ &(C\gamma^\nu T^b)_{\rho\sigma} u_\rho^{r_1}(\overrightarrow{q_1}) = \overline{v}_\rho^{r_1}(\overrightarrow{q_1})(\gamma^\nu T^b)_{\rho\sigma}, \end{aligned} \tag{6.56}$$

one can get

$$T_s(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k; E}) = g^2 \Delta_{\mu\nu}^{ab}(\overrightarrow{P}, E) \overline{u}^{s_1}(\overrightarrow{p_1}) \gamma^\mu T^a v^{s_2}(\overrightarrow{p_2}) \overline{v}^{r_1}(\overrightarrow{q_1}) \gamma^\nu T^b u^{r_2}(\overrightarrow{q_2}) \tag{6.57}$$

This just is the S-matrix element for the one-gluon exchange interaction taking place in the s-channel. It is easy to verify that the above matrix element is independent of the gauge parameter. Therefore, we only need to work in the Feynman gauge. In this gauge,

$$\Delta_{\mu\nu}^{ab}(\overrightarrow{P}, E) = \frac{\delta_{ab} g_{\mu\nu}}{E^2 - \overrightarrow{P}^2 + i\varepsilon} \tag{6.58}$$

With this propagator, as shown in Ref. [35], by the charge conjugation and Fierz transformation, Eq. (6.57) can be represented in the form

$$T_s(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k; E}) = \left(\frac{\omega_1 \omega_2}{m_1 m_2} \right)^{1/2} \varphi_{s_1}^+ \varphi_{s_2}^+ V_s^{(0)}(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k; E}) \varphi_{r_1} \varphi_{r_2} \tag{6.59}$$

where

$$V_s^{(0)}(\overrightarrow{P}, \overrightarrow{q}, \overrightarrow{k; E}) = -\frac{g^2 \widehat{C}_s \widehat{F}_s}{E^2 - \overrightarrow{P}^2 + i\varepsilon} \overline{U}(\overrightarrow{p_1}) \overline{U}(\overrightarrow{p_2}) \Gamma_{12} U(\overrightarrow{q_1}) U(\overrightarrow{q_2}) \tag{6.60}$$

is the interaction Hamiltonian occurring in the P-S equation in which the spinor is still defined in Eq. (5.1), $\widehat{C}_s$ is the color matrix

$$\widehat{C}_s = \frac{1}{24} (\lambda_1^a - \lambda_2^{a*})^2 \tag{6.61}$$

with λ_i^a being the Gell-Mann matrices, $\widehat{F}_s$ is the flavor matrix which has an expression for flavor SU(2) such that

$$\widehat{F}_s = \frac{1}{2} (1 - \overrightarrow{\tau_1} \cdot \overrightarrow{\tau_2}) \tag{6.62}$$

here $\overrightarrow{\tau_i}$ are isospin Pauli matrices and

$$\Gamma_{12} = -I_1 I_2 + \gamma_1^5 \gamma_2^5 - \frac{1}{2} \gamma_1^\mu \gamma_{2\mu} + \frac{1}{2} (\gamma_1^5 \gamma_1^\mu)(\gamma_2^5 \gamma_{2\mu}) \tag{6.63}$$

In the end, it is pointed out that since the matrix element of the color operator $\widehat{C}_s$ in the $q\overline{q}$ color singlet vanishes, the s-channel OGEK contributes nothing to the $q\overline{q}$ bound states. However, for many-quark-antiquark systems such as $\pi\pi$, $K\overline{K}$, πN, KN systems and etc., the contribution of the s-channel OGEK is not negligible and plays an important role to the interations taking place in those systems.[36- 38].

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7. Discussions and remarks

In this paper, the D-S equation satisfied by the $q\bar{q}$ bound states has been derived from QCD and, especially, the interaction kernel in the equation has been given two equivalent closed expressions which were respectively derived by making use of the equations of motion obeyed by the Green's functions and the irreducible decomposition of the Green's functions. Since the B-S equation is commonly viewed as the correct equation for the bound state problem, it is natural to ask what is the relation between the D-S equation and the B-S equation? As shown in Ref. [5], the B-S equation may be derived from the D-S equation. In fact, when applying the Lehmann representation in Eq. (2.36) to Eqs. (2.16) and (2.17), by the same procedure stated in section.2, one may obtain two D-S equations as follows

$$(i\partial_{x_1}-m_1)\chi_{P\varsigma}(x_1,x_2) = \int d^4y_1 d^4y_2 \widehat{K}_1(x_1,x_2;y_1,y_2)\chi_{P\varsigma}(y_1,y_2) \tag{7.1}$$

$$(i\partial_{x_2}-m_2)\chi_{P\varsigma}(x_1,x_2) = \int d^4y_1 d^4y_2 \widehat{K}_2(x_1,x_2;y_1,y_2)\chi_{P\varsigma}(y_1,y_2) \tag{7.2}$$

where

$$\widehat{K}_i(x_1,x_2;y_1,y_2) = \gamma_i^0 K_i(x_1,x_2;y_1,y_2), \ i=1,2 \tag{7.3}$$

in which $K_i(x_1,x_2;y_1,y_2)$ was given in Eq. (3.28). The D-S equations shown in Eqs. (2.38) and (2.39) may directly be written out from Eqs. (7.1) and (7.2).

Operating on Eq. (7.1) with $(i\partial_{x_2}-m_2)$ or on Eq. (7.2) with $(i\partial_{x_1}-m_1)$, we have

$$(i\partial_{x_1}-m_1)(i\partial_{x_2}-m_2)\chi_{P\varsigma}(x_1,x_2) = \int d^4y_1 d^4y_2 K_B(x_1,x_2;y_1,y_2)\chi_{P\varsigma}(y_1,y_2) \tag{7.4}$$

where

$$K_B(x_1,x_2;y_1,y_2) = (i\partial_{x_1}-m_1)\widehat{K}_2(x_1,x_2;y_1,y_2) = (i\partial_{x_2}-m_2)\widehat{K}_1(x_1,x_2;y_1,y_2) \tag{7.5}$$

is the B-S interaction kernel whose explicit expression was derived in Ref. [5]. Acting on Eq. (7.4) with the inverse of the operator $(i\partial_{x_1}-m_1)(i\partial_{x_2}-m_2)$, the B-S equation will be recast in an integral equation

$$\chi_{P\varsigma}(x_1,x_2) = \int d^4y_1 d^4y_2 d^4z_1 d^4z_2 S_F^{(0)}(x_1-z_1)S_F^{c(0)}(x_2-z_2)K_B(z_1,z_2;y_1,y_2)\chi_{P\varsigma}(y_1,y_2) \tag{7.6}$$

where $S_F^{(0)}(x_1-z_1)$ and $S_F^{c(0)}(x_2-z_2)$ are the free propagators of quark and antiquark respectively. In the momentum space, it becomes

$$\chi_{P\varsigma}(q) = S_F^{(0)}(p_1)S_F^{c(0)}(p_2)\int \frac{d^4k}{(2\pi)^4}K_B(P,q,k)\chi_{P\varsigma}(k) \tag{7.7}$$

Conversely, if we act on Eq. (7.6) with the operators $(i\partial_{x_2}-m_2)$ and $(i\partial_{x_1}-m_1)$ respectively, the D-S equations in Eqs. (7.1) and (7.2) will be recovered. This shows that to get the B-S equation, we need merely to consider the D-S equation. It should be noted that the four-dimensional B-S equation can not directly be transformed into the three-dimensional D-S equation. In order to obtain a three-dimensional equation from the four-dimensional B-S equation, it is necessary to introduce a certain constraint condition on the relative time (or relative energy) as was done in an approximate manner such as the instantaneous approximation or the quasipotential approaches [7-12].

As stated above, the four-dimensional D-S equation and the corresponding B-S equation can be derived from one another. But, this does not mean that the D-S equation and the B-S equation are fully equivalent to each other, similar to the Dirac equation and the K-G equation which can also be derived from each other. As seen from Eq. (7.4), the B-S equation is a kind of second-order differential equation in the position space. Therefore, a solution to the equation depends on not only the amplitude at time origin, but also the time-differential of the amplitude at

the time origin as in the case for K-G equation. This probably is the origin that causes the B-S equation to have the unphysical solutions of negative norm. In order to exclude the unphysical solutions, as mentioned before, the common procedure is to recast the four-dimensional B-S equation in a three-dimensional form by eliminating the relative time (or the relative energy) from the equation. Certainly, the three-dimensional equation, particularly, the exact version of the equation presented in the sections 2 and 3 is much convenient to use in solving the eigenvalue problem. However, since the three-dimensional equation loses the Lorentz-covariance of a relativistic dynamics, it is sometimes not suitable for carrying out extensive theoretical analyses, for instance, to perform the irreducible decomposition of the Green's functions contained in the kernel given in Eq. (3.30). The decomposition can readily be done in the four-dimensional form as shown in section 4. At this point, we may ask whether the relativistic bound state problem can be solved Lorentz-covariantly in the Minkowski space without occurrence of the unphysical solutions? The answer should be positive because the Lorentz-covariance of the equation implies that one may work in any Lorentz frame and gets the same result. Let us turn to the D-S equations shown in Eqs. (2.41) and (2.42) which are represented in the position space. Clearly, the equation in Eq.(2.42) is a first-order differential equation of Schrödinger-type. One may first solve this equation to get an amplitude which describes the evolution of the amplitude with the relative time τ. and then substitute this amplitude into Eq.(2.41) to solve the eigenvalue E and the amplitude $\chi_{P_\varsigma}(x)$. In solving these equations, we only need the initial conditions of the amplitude at the time origin without concerning the initial conditions of the time-differentials of the amplitude. Therefore, the unphysical solutions would not appear in this case. In this sense, we can say, the D-S equation derived in this paper, as it provides a new formulation of the relativistic equation for the two fermion bound system, gives a suitable prescription to solve the four-dimensional equation. Moreover, based on the relation denoted in Eq. (7.5), the B-S kernel may conveniently be evaluated from the D-S kernel. In comparison with the closed expression of the B-S kernel derived in Ref. [5], the D-S kernel shown in Eq. (3.30) is rather simpler. The main contribution to the D-S kernel is given by the Green's function $\mathcal{G}_{\mu\nu}^{ab}(x_i, y_j \mid x_1, x_2; y_1, y_2)$ written in Eq. (3.14). While, the B-S kernel concerns more complicated Green's functions such as [5]

$$
\begin{aligned}
&\mathcal{G}_{\mu\nu\lambda\tau}^{abcd}(x_1, x_2, y_1, y_2 \mid x_1, x_2; y_1, y_2) \\
&= \left\langle 0^+ \left| T\{N[\mathbf{A}_\mu^a(x_1)\mathbf{A}_\nu^b(x_2)\psi(x_1)\psi^c(x_2)]N[\mathbf{A}_\lambda^c(y_1)\mathbf{A}_\tau^d(y_2)\overline{\psi}(y_1)\overline{\psi}^c(y_2)]\right| 0^- \right\rangle
\end{aligned}
\tag{7.8}
$$

which gives the major contribution to the B-S kernel. In particular, in comparison of the four-dimensional kernel represented in Eqs. (3.30)-(3.33) with the three-dimensional counterpart written in Eqs. (3.40-)-(3.43), we see, there is an one-to-one correspondence between the both kernels. Therefore, to calculate the three-dimensional kernel, one may first calculate the four-dimensional one and then convert it to the three-dimensional form according the correspondence relation between the both of them. Since the four-dimensional D-S equation is Lorentz-covariant, its kernel can conveniently be analyzed and calculated by means of the familiar technique developed in the covariant quantum field theory.

In the end, we would like to address that unlike the Dyson-Schwinger equation [39, 40] which contains an infinite set of equations, the D-S equation derived in this paper is of a closed form with a closed expression of the kernel as given in section 3 or section 4. The kernel can easily be calculated by the perturbation method. For example, in the perturbative calculation of the kernel given in section 3 which contains only a few types of Green's functions, we only need the familiar perturbative expansions of the Green's functions without concerning the calculation of other more-point Green's functions as it is necessary to be done for the Dyson-Schwinger equation. Especially, each of the Green's functions can be represented in the form of functional integral and is possible to be evaluated by a nonperturbative method as suggested, for example, by the lattice gauge theory. Therefore, the expression of the kernel given in this paper provides a new formalism for exploring the QCD nonperturbative effect and the quark confinement which are important for the formation of a $q\overline{q}$ bound state. In the ordinary quark potential model [3, 41], the quark confinement is usually simulated by a linear potential which was suggested by the lattice computation of a Wilson loop and by the area law [42-44]. Obviously, this simulation is oversimplified. For the purpose of investigating the quark confinement, it is appropriate to start from the kernel given in this paper for the case that the quark and the antiquark have different flavors. In this case, all the Green's functions become the ordinary ones as shown in Eqs. (2.5), (2.13) and (3.11). Since the kernel derived in this paper contains all the interactions taking place in the bound state and includes the color-spin matrices $\Omega_i^{a\mu}$ defined in Eq. (2.23) in it, it is anticipated that a nonperturbative calculation of this kernel would give a sophisticated confining potential which includes not only its spatial form, but also its spin and color structures. This just is the advantage of the formalism of D-S equation presented in this paper.

8. Acknowledgment

9. Appendix A: Derivation of equations of motion satisfied by the Green's functions

This appendix is used to derive the equations of motion satisfied by the quark-antiquark two and four-point Green's functions. These equations may be derived from the following QCD generating functional [18].

$$Z[J, \overline{\eta}, \eta, \overline{\xi}, \xi] = \frac{1}{N} \int \mathcal{D}(A, \overline{\psi}, \psi, \overline{C}, C) e^{iI} \tag{A.1}$$

where

$$I = \int d^4x [\mathcal{L} + J^{a\mu} A_\mu^a + \overline{\eta}\psi + \overline{\psi}\eta + \overline{\xi}C + \overline{C}\xi] \tag{A.2}$$

in which $\mathcal{L}$ is the effective Lagrangian of QCD

$$\mathcal{L} = \overline{\psi}(i\partial - m + g\mathbf{A})\psi - \frac{1}{4}F^{a\mu\nu}F_{\mu\nu}^a - \frac{1}{2\alpha}(\partial^\mu A_\mu^a)^2 + \overline{C}^a \partial^\mu (D_\mu^{ab} C^b) \tag{A.3}$$

here $\mathbf{A} = \gamma^\mu T^a A_\mu^a$ with A_μ^a being the vector potentials of gluon fields,

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \tag{A.4}$$

are the strength tensors of the gluon field,

$$D_\mu^{ab} = \delta^{ab} \partial_\mu - g f^{abc} A_\mu^c \tag{A.5}$$

are the covariant derivatives, $\overline{C}^a, C^b$ represent the ghost fields, and $J^{a\mu}, \overline{\eta}, \eta, \overline{\xi}$ and ξ denote the external sources coupled to the gluon, quark and ghost fields respectively. By the charge conjugation transformations shown in Eq. (2.2) for the quark fields and in the following for the external sources

$$\eta^c = C\overline{\eta}^T, \overline{\eta}^c = -\eta^T C^{-1} \tag{A.6}$$

it is easy to prove the relation

$$\overline{\psi}(i\partial - m + g\mathbf{A})\psi + \overline{\eta}\psi + \overline{\psi}\eta = \overline{\psi}^c(i\partial - m + g\overline{\mathbf{A}})\psi^c + \overline{\eta}^c \psi^c + \overline{\psi}^c \eta^c \tag{A.7}$$

where $\overline{\mathbf{A}} = \gamma^\mu \overline{T}^a A_\mu^a$.

A. Equations of motion with respect to the coordinate x_1

Upon taking the functional derivative of the generating functional in Eq. (A.1) with respect to the field function $\overline{\psi}_\alpha(x_1)$ and considering

$$\frac{\delta Z}{\delta \overline{\psi}_\alpha(x_1)} = 0 \tag{A.8}$$

and

$$\frac{\delta I}{\delta \overline{\psi}_\alpha(x_1)} = \eta_\alpha(x_1) + [(i\partial_{x_1} - m_1)_{\alpha\gamma} + g\mathbf{A}(x_1)_{\alpha\gamma}]\psi_\gamma(x_1) \tag{A.9}$$

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it can be found that

$$\{\eta_\alpha(x_1) + [(i\partial_{x_1} - m_1)_{\alpha\gamma} + (\Gamma^{a\mu})_{\alpha\gamma}\frac{\delta}{i\delta J^{a\mu}(x_1)}]\frac{\delta}{i\delta\overline{\eta}_\gamma(x_1)}\}Z = 0 \tag{A.10}$$

where the fields $A^a_\mu(x_1)$ and $\psi_\gamma(x_1)$ have been replaced by the derivatives of the generating functional with respect to the sources $J^{a\mu}(x_1)$ and $\overline{\eta}_\gamma(x_1)$ and each of the subscripts α, β and γ marks the components of color, flavor and spinor. Differentiating Eq. (A.10) with respect to the source $\eta_\rho(y_1)$ and then setting all the sources to vanish, we obtain the equation satisfied by the quark propagator [18]

$$[(i\partial_{x_1} - m_1 + \Sigma)S_F]_{\alpha\rho}(x_1, y_1) = \delta_{\alpha\rho}\delta^4(x_1 - y_1) \tag{A.11}$$

where

$$\begin{aligned}(\Sigma S_F)_{\alpha\rho}(x_1, y_1) &\equiv \int d^4z_1 \Sigma(x_1, z_1)_{\alpha\gamma}S_F(z_1 - y_1)_{\gamma\rho} \\ &= (\Gamma^{a\mu})_{\alpha\gamma}\Lambda^a_\mu(x_1 \mid x_1, y_1)_{\gamma\beta}\end{aligned} \tag{A.12}$$

here $\Sigma(x_1, z_1)$ stands for the quark proper self-energy and $\Lambda^a_\mu(x_1 \mid x_1, y_1)_{\gamma\rho}$ was defined in the first equality of Eq. (3.11).

Let us turn to derive the equation of motion satisfied by the Green's function defined in Eq. (2.1). In doing this, we need first to derive the equations of motion obeyed by the Green's function defined in Eq. (2.5). By successively differentiating Eq. (A.10) with respect to the sources $\overline{\eta}^c_\beta(x_2), \eta_\rho(y_1)$ and $\eta^c_\sigma(y_2)$, noticing the equality in Eq. (A.7) and the following nonvanishing derivatives

$$\begin{aligned}\frac{\delta\eta^c_\alpha(x)}{\delta\overline{\eta}_\beta(y)} &= C_{\alpha\beta}\delta^4(x-y), \frac{\delta\overline{\eta}^c_\alpha(x)}{\delta\eta_\beta(y)} = (C^{-1})_{\alpha\beta}\delta^4(x-y), \\ \frac{\delta\overline{\eta}_\alpha(x)}{\delta\eta^c_\beta(y)} &= (C^{-1})_{\alpha\beta}\delta^4(x-y), \frac{\delta\eta_\alpha(x)}{\delta\overline{\eta}^c_\beta(y)} = C_{\alpha\beta}\delta^4(x-y)\end{aligned} \tag{A.13}$$

we have

$$\begin{aligned}&\{C_{\alpha\beta}\delta^4(x_1 - x_2)\frac{\delta}{i^3\delta\eta_\rho(y_1)\delta\eta^c_\sigma(y_2)} + \delta_{\alpha\rho}\delta^4(x_1 - y_1)\frac{\delta^2}{i\delta\overline{\eta}^c_\beta(x_2)\delta\eta^c_\sigma(y_2)} \\ &+\eta_\alpha(x_1)\frac{\delta^3}{i\delta\overline{\eta}^c_\beta(x_2)\delta\eta_\rho(y_1)\delta\eta^c_\sigma(y_2)} - [(i\partial_{x_1} - m_1)_{\alpha\gamma} + (\Gamma^{a\mu})_{\alpha\gamma}\frac{\delta}{i\delta J^{a\mu}(x_1)}] \\ &\times\frac{\delta^4}{\delta\overline{\eta}_\gamma(x_1)\delta\overline{\eta}^c_\beta(x_2)\delta\eta_\rho(y_1)\delta\eta^c_\sigma(y_2)}\}Z = 0\end{aligned} \tag{A.14}$$

When all the sources are set to be zero, one immediately derives Eq. (2.10) from Eq. (A.14).

It is noted that the first equation in Eq. (2.14) may directly derived from Eq. (A.11) by the charge conjugation transformation represented in Eq. (2.2) or by differentiating Eq. (A.10) with respect to the source $\overline{\eta}^c_\beta(x_2)$.

B. Equations of motion with respect to the coordinate x_2

When taking the derivative of the generating functional in Eq. (A.1) with respect to the field variable $\overline{\psi}^c_\beta(x_2)$ and noticing the relation in Eq. (A.7), by the same procedure as described in Eqs. (A.8)-(A.10), one may obtain

$$\{\eta^c_\beta(x_2) + [(i\partial_{x_2} - m_2)_{\beta\lambda} + (\overline{\Gamma}^{b\nu})_{\beta\lambda}\frac{\delta}{i\delta J^{b\nu}(x_2)}]\frac{\delta}{i\delta\overline{\eta}^c_\lambda(x_2)}\}Z = 0 \tag{A.15}$$

Differentiating Eq. (A.15) with respect to $\eta^c_\sigma(y_2)$ and then setting all the sources to be zero, one can get the equation for the antiquark propagator

$$[(i\partial_{x_2} - m_2 + \Sigma^c)S^c_F]_{\beta\sigma}(x_2, y_2) = \delta_{\beta\sigma}\delta^4(x_2 - y_2) \tag{A.16}$$

where

$$\begin{aligned}(\Sigma^c S^c_F)_{\beta\sigma}(x_2, y_2) &\equiv \int d^4z_2 \Sigma^c(x_2, z_2)_{\beta\lambda}S^c_F(z_2 - y_2)_{\lambda\sigma} \\ &= (\overline{\Gamma}^{b\nu})_{\beta\lambda}\Lambda^{cb}_\nu(x_2 \mid x_2, y_2)_{\lambda\sigma}\end{aligned} \tag{A.17}$$

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here $\Sigma^c(x_2, z_2)$ is the antiquark proper self-energy and $\Lambda_\nu^{\mathbf{cb}}(x_2 \mid x_2, y_2)_{\lambda\sigma}$ was represented in the second equality of Eq. (3.12).

Similarly, Upon differentiating Eq. (A.15) with respect to the sources $\overline{\eta}_\alpha(x_1), \eta_\rho(y_1)$ and $\eta_\sigma^c(y_2)$, one gets

$$\begin{aligned}
&\{C_{\alpha\beta}\delta^4(x_1 - x_2)\tfrac{\delta}{i^3\delta\eta_\rho(y_1)\delta\eta_\sigma^c(y_2)} + \delta_{\beta\sigma}\delta^4(x_2 - y_2)\tfrac{\delta^2}{i\delta\overline{\eta}_\alpha(x_1)\delta\eta_\rho(y_2)} \\
&+\eta_\beta^c(x_2)\tfrac{\delta^3}{i\delta\overline{\eta}_\alpha(x_1)\delta\eta_\rho(y_1)\delta\eta_\sigma^c(y_2)} - [(i\partial_{x_2} - m_2)_{\beta\lambda} + (\overline{\Gamma}^{b\nu})_{\beta\lambda}\tfrac{\delta}{i\delta J^{b\nu}(x_2)}] \\
&\times\tfrac{\delta^4}{\delta\overline{\eta}_\alpha(x_1)\delta\overline{\eta}_\lambda^c(x_2)\delta\eta_\rho(y_1)\delta\eta_\sigma^c(y_2)}\}Z = 0
\end{aligned} \tag{A.18}$$

Setting all the sources to vanish, we directly obtain Eq. (2.11) from the above equation.

It is mentioned that the second equation in Eq. (2.14) may directly be derived from Eq. (A.16) by the charge conjugation transformation or by differentiating Eq. (A.15) with respect to the source $\overline{\eta}_\alpha(x_1)$.

C. Equations of motion with respect to the coordinate y_1

By taking the derivative of the generating functional in Eq. (A.1) with respect to the field variable $\psi_\rho(y_1)$, following the same procedure as deriving Eq. (A.10), one may get

$$\{\overline{\eta}_\rho(y_1) + \tfrac{\delta}{i\delta\eta_\tau(y_1)}[(i\overleftarrow{\partial}_{y_1} + m_1)_{\tau\rho} - (\Gamma^{a\mu})_{\tau\rho}\tfrac{\delta}{i\delta J^{a\mu}(y_1)}]\}Z = 0 \tag{A.19}$$

On differentiating the above equation with respect to $\overline{\eta}_\alpha(x_1)$ and then turning off all the sources, we arrive at

$$[S_F(i\overleftarrow{\partial}_{y_1} + m_1 - \Sigma)]_{\alpha\rho}(x_1, y_1) = -\delta_{\alpha\rho}\delta^4(x_1 - y_1) \tag{A.20}$$

where

$$\begin{aligned}
(S_F\Sigma)_{\alpha\rho}(x_1, y_1) &\equiv \int d^4z_1 S_F(x_1 - z_1)_{\alpha\tau}\Sigma(z_1, y_1)_{\tau\rho} \\
&= \Lambda_\mu^a(y_1 \mid x_1, y_1)_{\alpha\tau}(\Gamma^{a\mu})_{\tau\rho}
\end{aligned} \tag{A.21}$$

If we differentiate Eq. (A.19) with respect to $\eta_\sigma^c(y_2)$, after letting the sources to be vanishing, we get an equation satisfied by the propagator $\overline{S}_F^*(y_1 - y_2)$ as written in the first equation in Eq. (3.3).

Now let us differentiate Eq. (A.19) with respect to $\overline{\eta}_\alpha(x_1), \overline{\eta}_\beta^c(x_2)$ and $\eta_\sigma^c(y_2)$ and then set all the sources but the source J to be zero. By these operations, we get

$$\begin{aligned}
&G(x_1, x_2; y_1, y_2)_{\alpha\beta\tau\sigma}^J(i\overleftarrow{\partial}_{y_1} + m_1)_{\tau\rho} = -\delta_{\alpha\rho}\delta^4(x_1 - y_1)S_F^c(x_2 - y_2)_{\beta\sigma}^J \\
&-C_{\rho\sigma}\delta^4(y_1 - y_2)S_F^*(x_1 - x_2)_{\alpha\beta}^J + G_\mu^a(y_1 \mid x_1, x_2; y_1, y_2)_{\alpha\beta\tau\sigma}^J(\Gamma^{a\mu})_{\tau\rho}
\end{aligned} \tag{A.22}$$

where

$$G(x_1, x_2; y_1, y_2)^J = \frac{\delta^4 Z[J, \overline{\eta}, \eta, \overline{\xi}, \xi]}{\delta\overline{\eta}(x_1)\delta\overline{\eta}^c(x_2)\delta\eta(y_1)\delta\eta^c(y_2)}\mid_{\overline{\eta}=\overline{\eta}^c=\eta=\eta^c=0} \tag{A.23}$$

$$S_F^c(x_2 - y_2)_{\beta\sigma}^J = \frac{\delta^2 Z[J, \overline{\eta}, \eta, \overline{\xi}, \xi]}{i\delta\overline{\eta}_\beta^c(x_2)\delta\eta_\sigma^c(y_2)}\mid_{\overline{\eta}=\eta=\overline{\xi}=\xi=0} \tag{A.24}$$

$$S_F^*(x_1 - x_2)_{\alpha\beta}^J == \frac{\delta^2 Z[J, \overline{\eta}, \eta, \overline{\xi}, \xi]}{i^3\delta\overline{\eta}_\alpha(x_1)\delta\overline{\eta}_\beta^c(x_2)}\mid_{\overline{\eta}=\eta=\overline{\xi}=\xi=0} \tag{A.25}$$

and

$$G_\mu^a(y_i \mid x_1, x_2; y_1, y_2)^J = \frac{\delta}{i\delta J^{a\mu}(y_i)}G(x_1, x_2; y_1, y_2)^J, i = 1, 2 \tag{A.26}$$

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Once we set $J = 0$, Eq. (A.22) will give rise to Eq. (3.1). Furthermore, if we differentiate Eq. (A.22) with respect to $J^{a\mu}(x_i)$ and subsequently set $J = 0$, noticing

$$G_{\mu\nu}^{ab}(x_i, y_j \mid x_1, x_2; y_1, y_2) = \frac{\delta}{i\delta J^{a\mu}(x_i) i\delta J^{b\nu}(y_j)} G(x_1, x_2; y_1, y_2)^J \mid_{J=0} \qquad (A.27)$$

the equations in Eq. (3.8) will immediately be derived.

D. Equations of motion with respect to the coordinate y_2

To derive the equations of motion with respect to y_2, we need to differentiate the generating functional with respect to the field $\psi_\sigma^c(y_2)$. By the same procedure as formulated in Eqs. (A.8)-(A.10), we get

$$\{\overline{\eta}_\sigma^c(y_2) + \frac{\delta}{i\delta\eta_\delta^c(y_2)}[(i\overleftarrow{\partial}_{y_2} + m_2)_{\delta\sigma} - (\overline{\Gamma}^{b\nu})_{\delta\sigma}\frac{\delta}{i\delta J^{b\nu}(y_2)}]\}Z = 0 \qquad (A.28)$$

Then, the differentiation of the above equation with respect to $\overline{\eta}_\beta^c(x_2)$ with setting all the sources to vanish subsequently will lead us to

$$[S_F^c(i\overleftarrow{\partial}_{y_2} + m_2 - \Sigma^c)]_{\beta\sigma}(x_2, y_2) = -\delta_{\beta\sigma}\delta^4(x_2 - y_2) \qquad (A.29)$$

where

$$(S_F^c\Sigma^c)_{\beta\sigma}(x_2, y_2) \equiv \int d^4z_2 S_F^c(x_2 - z_2)_{\beta\delta}\Sigma^c(z_2, y_2)_{\delta\sigma}$$
$$= \Lambda_\nu^{cb}(y_2 \mid x_2, y_2)_{\beta\delta}(\overline{\Gamma}^{b\nu})_{\delta\sigma} \qquad (A.30)$$

Upon differentiating Eq. (A.28) with respect to $\eta_\rho(y_1)$ and then turning off all the sources, one can obtain the second equation in Eq. (3.3) for the propagator $\overline{S}_F^*(y_1 - y_2)$

Let us differentiate Eq. (A.28) with respect to $\overline{\eta}_\alpha(x_1), \overline{\eta}_\beta^c(x_2)$ and $\eta_\rho(y_1)$ and set all the sources except for the J to vanish. As a result, we get

$$G(x_1, x_2; y_1, y_2)_{\alpha\beta\rho\delta}^J(i\overleftarrow{\partial}_{y_2} + m_2)_{\delta\sigma} = -\delta_{\beta\sigma}\delta^4(x_2 - y_2)S_F(x_1 - y_1)_{\alpha\rho}^J$$
$$-C_{\rho\sigma}\delta^4(y_1 - y_2)S_F^*(x_1 - x_2)_{\alpha\beta}^J + G_\nu^b(y_2 \mid x_1, x_2; y_1, y_2)_{\alpha\beta\rho\delta}^J(\overline{\Gamma}^{b\nu})_{\delta\sigma} \qquad (A.31)$$

where

$$S_F(x_1 - y_1)_{\gamma\rho}^J = \frac{\delta^2 Z[J, \overline{\eta}, \eta, \overline{\xi}, \xi]}{i\delta\overline{\eta}_\gamma(x_1)\delta\eta_\rho(y_1)} \mid_{\overline{\eta}=\eta=\overline{\xi}=\xi=0} \qquad (A.32)$$

and the other Green's functions in the presence of source J were defined before. When we set $J = 0$, Eq. (A.31) straightforwardly yields Eq. (3.2). Finally, on differentiating Eq. (A.31) with respect to $J^{a\mu}(x_i)$ and subsequently setting $J = 0$, Eq. (3.9) will be derived.

10. Appendix B: One-particle irreducible decompositions of the connected Green's functions

Let us begin with the relation between the generating functional for full Green's functions $Z[J, \overline{\eta}, \eta, \overline{\xi}, \xi]$ and the one for connected Green's functions $W[J, \overline{\eta}, \eta, \overline{\xi}, \xi]$ [18, 33]

$$Z[J, \overline{\eta}, \eta, \overline{\xi}, \xi] = \exp\{iW[J, \overline{\eta}, \eta, \overline{\xi}, \xi]\} \qquad (B.1)$$

Taking the derivatives of Eq. (B.1) with respect to the sources $\overline{\eta}(x_1), \overline{\eta}^c(x_2), \eta(y_1)$ and $\eta^c(y_2)$ and then setting all the sources except for the source J to be zero, one may obtain the following decomposition

$$G(x_1,x_2;y_1,y_2)^J = G_c(x_1,x_2;y_1,y_2)^J + S_F(x_1 - y_1)^J S_F^c(x_2 - y_2)^J$$
$$-S_F^*(x_1 - x_2)^J \overline{S}_F^*(y_1 - y_2)^J \tag{B.2}$$

where $G(x_1,x_2;y_1,y_2)^J$, $S_F^c(x_2 - y_2)^J$, $S_F^*(x_1 - x_2)^J$ and $S_F(x_1 - y_1)^J$ were defined in Eqs. (A.23)-(A.25) and (A.32) respectively, while, $G_c(x_1,x_2;y_1,y_2)^J$ and $\overline{S}_F^*(y_1 - y_2)^J$ are defined by

$$G_c(x_1,x_2;y_1,y_2)^J = i\frac{\delta^4 W[J,\overline{\eta},\eta,\overline{\xi},\xi]}{\delta\overline{\eta}(x_1)\delta\overline{\eta}^c(x_2)\delta\eta(y_1)\delta\eta^c(y_2)} \mid_{\overline{\eta}=\overline{\eta}^c=\eta=\eta^c=0}, \tag{B.3}$$

and

$$\overline{S}_F^*(y_1 - y_2)^J = \frac{\delta^2 W[J,\overline{\eta},\eta,\overline{\xi},\xi]}{i^2\delta\eta(y_1)\delta\eta^c(y_2)} \mid_{\overline{\eta}=\eta=\overline{\xi}=\xi=0} \tag{B.4}$$

When we set $J = 0$, Eq. (B.2) will go over to the decomposition shown in Eq. (4.1). Differentiating Eq. (B.2) with respect to the source $J^{a\mu}(x_i)$, we have

$$G_\mu^a(x_i \mid x_1,x_2;y_1,y_2)^J = G_{c\mu}^a(x_i \mid x_1,x_2;y_1,y_2)^J$$
$$+\Lambda_\mu^a(x_i \mid x_1;y_1)^J S_F^c(x_1 - y_1)^J + S_F(x_1 - y_1)^J \Lambda_\mu^{ca}(x_i \mid x_2;y_2)^J$$
$$-\Lambda_\mu^{a*}(x_i \mid x_1,x_2)^J \overline{S}_F^*(y_1 - y_2)^J - S_F^*(x_1 - x_2)^J \overline{\Lambda}_\mu^{a*}(x_i \mid y_1,y_2)^J \tag{B.5}$$

where $G_\mu^a(x_i \mid x_1,x_2;y_1,y_2)^J$ was defined in Eq. (A.26) with y_i being replaced by x_i and the other functions are defined by

$$G_{c\mu}^a(x_i \mid x_1,x_2;y_1,y_2)^J = \frac{\delta}{i\delta J^{a\mu}(x_i)}G_c(x_1,x_2;y_1,y_2)^J \tag{B.6}$$

$$\Lambda_\mu^a(x_i \mid x_1;y_1)^J = \frac{\delta}{i\delta J^{a\mu}(x_i)}S_F(x_1 - y_1)^J \tag{B.7}$$

$$\Lambda_\mu^{ca}(x_i \mid x_1;y_1)^J = \frac{\delta}{i\delta J^{a\mu}(x_i)}S_F^c(x_1 - y_1)^J \tag{B.8}$$

$$\Lambda_\mu^{a*}(x_i \mid x_1,x_2)^J = \frac{\delta}{i\delta J^{a\mu}(x_i)}S_F^*(x_1 - x_2)^J \tag{B.9}$$

and

$$\overline{\Lambda}_\mu^{a*}(x_i \mid y_1,y_2)^J = \frac{\delta}{i\delta J^{a\mu}(x_i)}\overline{S}_F^*(y_1 - y_2)^J \tag{B.10}$$

Upon setting $J = 0$, Eq. (B.5) immediately gives rise to the decomposition in Eq. (4.3).

Now, let us proceed to carry out one-particle-irreducible decompositions of the connected Green's functions on the RHS of Eq. (4.4). The decompositions are easily performed with the help of the Legendre transformation which is described by the relation between the generating functional of proper vertices Γ and the one for connected Green's functions W [18, 33]

$$\Gamma[A_\mu^a,\overline{\psi},\psi,\overline{C}^a,C^a] = W[J,\overline{\eta},\eta,\overline{\xi},\xi] - \int d^4x[J^{a\mu}A_\mu^a + \overline{\eta}\psi + \overline{\psi}\eta + \overline{\xi}C + \overline{C}\xi] \tag{B.11}$$

and the relations between the field functions and the external sources

$$\psi(x) = \frac{\delta W}{\delta\overline{\eta}(x)}, \overline{\psi}(x) = -\frac{\delta W}{\delta\eta(x)}, A_\mu^a(x) = \frac{\delta W}{\delta J^{a\mu}(x)}, C^a(x) = \frac{\delta W}{\delta\overline{\xi}^a(x)}, \overline{C}^a(x) = -\frac{\delta W}{\delta\xi^a(x)} \tag{B.12}$$

$$\eta(x) = -\frac{\delta\Gamma}{\delta\overline{\psi}(x)}, \overline{\eta}(x) = \frac{\delta\Gamma}{\delta\psi(x)}, J_\mu^a(x) = -\frac{\delta\Gamma}{\delta A^{a\mu}(x)}, \xi^a(x) = -\frac{\delta\Gamma}{\delta\overline{C}^a(x)}, \overline{\xi}^a(x) = \frac{\delta\Gamma}{\delta C^a(x)} \tag{B.13}$$

where the field functions in Eq. (B.12) are all functionals of the external sources in Eq. (B.13) and, simultaneously, the sources in Eq. (B.13) are all functionals of the field functions in Eq. (B.12).

Taking the derivative of the both sides of the first equality in Eq. (B.12) with respect to $\psi(y)$ and employing the first relation in Eq. (B.13), one may get

$$\int d^4z \frac{\delta^2\Gamma}{\delta\psi(y)\delta\overline{\psi}(z)} \frac{\delta^2 W}{\delta\eta(z)\delta\overline{\eta}(x)} = \int d^4z \frac{\delta^2 W}{\delta\overline{\eta}(x)\delta\eta(z)} \frac{\delta^2\Gamma}{\delta\overline{\psi}(z)\delta\psi(y)} = -\delta^4(x-y) \tag{B.14}$$

where we only keep the term on the RHS of Eq. (B.14) which is nonvanishing when the sources are set to vanish. In order to find the one-particle-irreducible decomposition for the quark-gluon three-point Green's functions, one may differentiate Eq. (B.14) with respect to the source $J^{a\mu}(x_i)$ and then using Eq. (B.14) once again. By this procedure, it can be derived that

$$\frac{\delta^3 W}{\delta J^{a\mu}(x_i)\delta\overline{\eta}(x_j)\delta\eta(y_k)} = \int d^4z d^4u_1 d^4u_2 \frac{\delta^2 W}{\delta J^{a\mu}(x_i)\delta J^{b\nu}(u_1)} \frac{\delta^2 W}{\delta\overline{\eta}(x_j)\delta\eta(u_2)}$$
$$\frac{\delta^3\Gamma}{\delta A_\nu^b(u_1)\delta\overline{\psi}(u_2)\psi(z)} \frac{\delta^2 W}{\delta\overline{\eta}(z)\delta\eta(y_k)} \tag{B.15}$$

where the coordinates in Eq. (B.14) have been appropriately changed. When all the sources are set to be zero, noticing the definitions given in Eq. (A.32) where the Z is replaced by iW and in Eq. (B.7) as well as

$$\Delta_{\mu\nu}^{ab}(x_i - y_j) = \frac{\delta^2 W}{i^2\delta J^{a\mu}(x_i)\delta J^{b\nu}(y_j)}\Big|_{J=0} \tag{B.16}$$

$$\Gamma^{b\nu}(u_1 \mid u_2, z) = i\frac{\delta^3\Gamma}{\delta A_\nu^b(u_1)\delta\overline{\psi}(u_2)\delta\psi(z)}\Big|_{A=\overline{\psi}=\psi=0} \tag{B.17}$$

the decomposition shown in Eqs. (4.5) and (4.6) straightforwardly follows from Eq. (B.15). Analogously, if we replace $\overline{\eta}(x_j)$ and $\eta(y_k)$ by $\overline{\eta}^c(x_j)$ and $\eta^c(y_k)$ in Eq. (B.15) and noticing

$$\Gamma_c^{b\nu}(u_1 \mid u_2, z) = i\frac{\delta^3\Gamma}{\delta A_\nu^b(u_1)\delta\overline{\psi}^c(u_2)\delta\psi^c(z)}\Big|_{A=\overline{\psi}=\psi=0} \tag{B.18}$$

the decomposition shown in Eq. (4.8) and (4.9) will be derived. This decomposition may also be derived from Eq. (B.15) by the charge conjugation transformation for the quark fields. By this transformation, one may readily derive from Eq. (B.15) the decomposition denoted in Eq. (4.15) in which the gluon-quark-antiquark vertex is defined by

$$\overline{\Gamma}^{b\nu*}(z \mid z_1, z_2) = i\frac{\delta^3\Gamma}{\delta A_\nu^b(z)\delta\psi(z_1)\delta\psi^c(z_2)}\Big|_{A=\overline{\psi}=\psi=0} \tag{B.19}$$

The one-particle-irreducible decomposition of the connected Green's function $G_c(x_1, x_2; y_1, y_2)$ can be derived by the same procedure as obtaining Eq. (B.15). On differentiating Eq. (B.14) with respect to $\overline{\eta}^c(x_2)$ and $\eta^c(y_2)$ and setting all the sources but the source J to vanish, one may obtain

$$G_c(x_1, x_2; y_1, y_2)^J = \int \prod_{i=1}^2 d^4u_i d^4v_i S_F(x_1 - u_1)^J S_F^c(x_2 - u_2)^J$$
$$\times \Gamma(u_1, u_2; v_1, v_2)^J S_F(v_1 - y_1)^J S_F^c(v_2 - y_2)^J \tag{B.20}$$

where the four-point connected Green's function and the propagators given in the presence of the sources were defined before and the function $\Gamma(u_1, u_2; v_1, v_2)^J$ is formally the same as that defined in Eqs. (4.19)-(4.21). When the source J is turned off, Eq. (B.20) directly goes over to the decomposition in Eq. (4.18) with the vertices in Eqs. (4.19)-(4.21) being defined in Eqs. (B.17)-(B.19) and in the following

$$\Gamma^{b\nu*}(x_1 \mid u_1, v_1) = i \frac{\delta^3 \Gamma}{\delta A_\nu^b(x_1) \delta \overline{\psi}(u_1) \delta \overline{\psi}^c(v_2)} \mid_{A=\overline{\psi}=\psi=0} \tag{B.21}$$

which is the charge conjugate to the vertex $\overline{\Gamma}^{b\nu*}(z \mid z_1, z_2)$ as well as

$$\Gamma_3(u_1, u_2; v_1, v_2) = i \frac{\delta^4 \Gamma}{\delta \overline{\psi}(u_1) \delta \overline{\psi}^c(u_2) \delta \psi(v_1) \delta \psi^c(v_2)} \mid_{\overline{\psi}=\psi=\overline{\psi}^c=\psi^c=0} \tag{B.22}$$

which is the quark-antiquark four-line proper vertex. It is emphasized here that the decomposition of the function $G_c(x_1, x_2; y_1, y_2)$ in the absence of the source J has the same form as that given in the presence of J. This is because the Green's function is defined only by the differentials with respect to the fermion fields as indicated in Eq. (B.3).

The one-particle irreducible decomposition of the Green's function $G_{c\mu}^a(x_i \mid x_1, x_2; y_1, y_2)$ may be derived by starting from the expression given in Eq. (B.15) with $j, k = 1$. By differentiating the both sides of Eq. (B.15) with respect to the sources $\overline{\eta}^c(x_2)$ and $\eta^c(y_2)$ and then turning off all the external sources, one may obtain the decomposition of the function $G_{c\mu}^a(x_i \mid x_1, x_2; y_1, y_2)$ as shown in Eqs. (4.22)-(4.25). Alternatively, the decomposition may also be obtained by starting with the expression written in Eq. (B.20). Substituting Eq. (B.20) into Eq. (B.6), then completing the differentiation with respect to the source $J^{a\mu}(x_i)$ and finally setting the source to vanish, one may also derive the irreducible decomposition of the function $G_{c\mu}^a(x_i \mid x_1, x_2; y_1, y_2)$. In doing this, it is necessary to perform the differentiations of the fermion propagators with respect to the source $J^{a\mu}(x_i)$ as shown in Eqs. (B.7)-(B.10) and use their decompositions presented in Eqs. (4.5)-(4.9). In addition, we need to carry out the differentiations of the gluon propagator and some vertices with respect to the source $J^{a\mu}(x_i)$ as shown below. For the gluon propagator defined in Eq. (B.16), from its representation in presence of the external source J ,in the same way as deriving the decomposition represented in Eqs. (B.15), (4.5) and (4.6), one may obtain the one-particle irreducible decomposition of the gluon three-point Green's function as follows:

$$\Lambda_{\mu\rho\sigma}^{acd}(x_i, z_1, z_2) = \frac{\delta}{\delta J^{a\mu}(x_i))} \Delta_{\rho\sigma}^{cd}(z_1 - z_2)^J \mid_{J=0} = \int d^4 z D_{\mu\nu}^{ab}(x_i - z) \Pi_{\rho\sigma}^{bcd,\nu}(z, z_1, z_2) \tag{B.23}$$

where $D_{\mu\nu}^{ab}(x_i - z) = i\Delta_{\mu\nu}^{ab}(x_i - z)$ and $\Pi_{\rho\sigma}^{bcd,\nu}(z, z_1, z_2)$ was represented in Eq. (4.32) with the gluon three-line proper vertex defined by

$$\Gamma_{bcd}^{\nu\rho\sigma}(z, u_1, u_2) = i \frac{\delta^3 \Gamma}{\delta A_\nu^b(z) \delta A_\rho^c(u_1) \delta A_\sigma^d(u_2)} \mid_{A=0} \tag{B.24}$$

For a proper vertex $\Gamma_\alpha(z_1, z_2, \cdots)$ with α marking the other indices, its derivative with respect to the source $J^{a\mu}(x_i)$ can be represented as

$$\frac{\delta}{i\delta J^{a\mu}(x_i))} \Gamma_\alpha(z_1, z_2, \cdots)^J \mid_{J=0} = \int d^4 z D_{\mu\nu}^{ab}(x_i - z) \Gamma_\alpha^{b\nu}(z, z_1, z_2, \cdots) \tag{B.25}$$

where

$$\Gamma_\alpha^{b\nu}(z, z_1, z_2, \cdots) = \frac{\delta}{\delta A_\nu^b(z)} \Gamma_\alpha(z_1, z_2, \cdots)^J \mid_{J=0} \tag{B.26}$$

According to the procedure stated above, it is not difficult to derive the expressions described in Eqs. (4.30)-(4.33). In the expressions, the vertices are defined as follows:

$$\Gamma^{a\mu}(x_i \mid u_1, u_2; v_1, v_2) = \frac{\delta \Gamma(u_1, u_2; v_1, v_2)^J}{i\delta J^{a\mu}(x_i)} \mid_{J=0} \tag{B.27}$$

$$\Gamma_{\nu\lambda}^{bc}(z, z_1 \mid u_1, v_1) = i \frac{\delta^4 \Gamma}{\delta A^{b\nu}(z) \delta A^{c\lambda}(z_1) \delta \overline{\psi}(u_1) \delta \psi(v_1)} \mid_{A=\overline{\psi}=\psi=0} \tag{B.28}$$

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$$\Gamma^{bc}_{\mathbf{c}\nu\lambda}(z,z_1 \mid u_1,v_1) = i\frac{\delta^4\Gamma}{\delta A^{b\nu}(z)\delta A^{c\lambda}(z_1)\delta\overline{\psi}^c(u_1)\delta\psi^c(v_1)} \mid_{A=\overline{\psi}^c=\psi^c=0} \tag{B.29}$$

$$\Gamma^{bc*}_{\nu\lambda}(z,z_1 \mid u_1,v_1) = i\frac{\delta^4\Gamma}{\delta A^{b\nu}(z)\delta A^{c\lambda}(z_1)\delta\overline{\psi}(u_1)\delta\overline{\psi}^c(v_1)} \mid_{A=\overline{\psi}=\overline{\psi}^c=0} \tag{B.30}$$

$$\overline{\Gamma}^{bc*}_{\mathbf{c}\nu\lambda}(z,z_1 \mid u_1,v_1) = i\frac{\delta^4\Gamma}{\delta A^{b\nu}(z)\delta A^{c\lambda}(z_1)\delta\psi(u_1)\delta\psi^c(v_1)} \mid_{A=\psi=\psi^c=0} \tag{B.31}$$

Particularly, by the following differentiation

$$\Gamma^{a\mu}_3(x_i \mid u_1,u_2;v_1,v_2) = \frac{\delta\Gamma_3(u_1,u_2;v_1,v_2)^J}{i\delta J^{a\mu}(x_i)} \mid_{J=0} \tag{B.32}$$

it is easy to give the expression in Eq. (4.34) in which

$$\widehat{\Gamma}^{b\nu}(z \mid u_1,u_2;v_1,v_2) = i\frac{\delta^4\Gamma}{\delta A^{b\nu}(z)\delta\overline{\psi}(u_1)\delta\overline{\psi}^c(u_2)\delta\psi(v_1)\delta\psi^c(v_2)} \mid_{A=\overline{\psi}=\psi=\overline{\psi}^c=\psi^c=0} \tag{B.33}$$

is the gluon-quark-antiquark five-line proper vertex.

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